

Worked Circuits IV: Sinusoids, Resonance and Complete Response

Ordinary Differential Equations for Electrical Engineers — Ch. 15 of 15

1. RC circuit switched onto a sinusoidal source at $t = 0$

Switching a sinusoid onto an RC circuit produces a sinusoidal forced response and an exponential transient. Phasor analysis gives only the surviving steady-state sinusoid; the complete initial-value problem needs the transient too.

ODE

For the input $V_m \cos \omega t \, u(t)$, the complete response is $v_C = v_{ss} + K e^{-t/RC}$:

$$RC \, v_C' + v_C = V_m \cos \omega t$$

Steady-state sinusoid

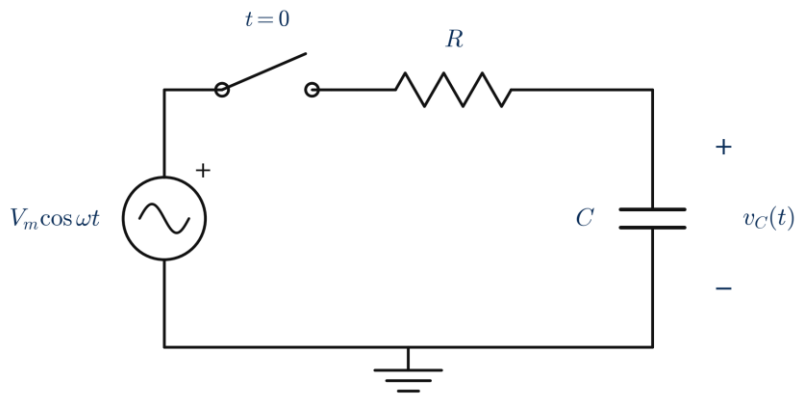
The frequency response gives the amplitude and phase of the forced term:

$$H(j\omega) = \frac{1}{1 + j\omega RC}, \quad |H| = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \quad \phi = -\tan^{-1}(\omega RC)$$

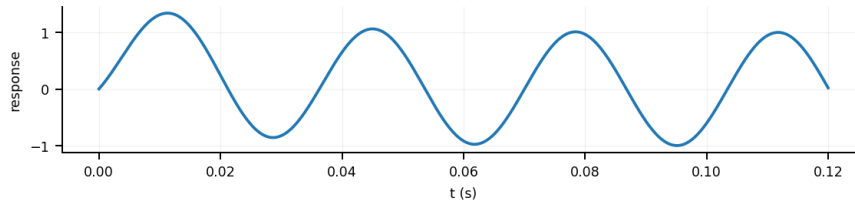
$$v_{ss} = V_m |H| \cos(\omega t + \phi)$$

Transient coefficient

Evaluate the steady-state sinusoid at $t = 0$ and use $v_C(0^+) = v_C(0^-)$ to find K . The transient exists only to make the complete solution satisfy the initial condition.



RC circuit switched onto a sinusoidal source at $t = 0$.



Worked example — zero initial voltage

$R = 1 \text{ k}\Omega$, $C = 100 \text{ }\mu\text{F}$, $f = 10 \text{ Hz}$ and $V_m = 5 \text{ V}$, so $\tau = 0.1 \text{ s}$ and $\omega\tau = 6.283$. Then $|H| = 0.1572$ and $\phi = -80.96^\circ$:

$$v_{ss} = 0.786 \cos(62.83t - 80.96^\circ) \text{ V}$$

At $t = 0$, $v_{ss} = 0.124 \text{ V}$, so $K = -0.124 \text{ V}$ enforces $v_c(0) = 0$.

2. RL circuit switched onto a sinusoidal source at $t = 0$

The current in a series RL circuit is also a sinusoidal forced term plus an exponential natural term, and current continuity fixes the transient amplitude.

ODE and frequency response

$L i' + Ri = V_m \cos \omega t$. The current-per-voltage transfer has magnitude $1/\sqrt{R^2 + (\omega L)^2}$ and phase $-\tan^{-1}(\omega L/R)$:

$$\frac{I}{V} = \frac{1}{R + j\omega L}, \quad i_{ss} = I_m \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

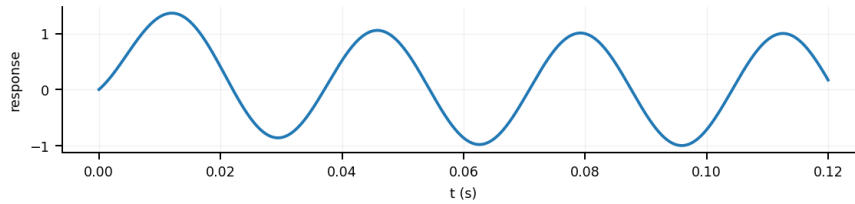
Complete response

If $i(0^-) = 0$, then $K = -i_{ss}(0)$:

$$i = i_{ss} + K e^{-tR/L}$$

Physical interpretation

At high frequency the inductive reactance is large, so the current is small and lags the voltage strongly. At low frequency the resistor dominates, and the current approaches V/R with little phase lag.



Worked example — sinusoidal current plus transient

$R = 20\ \Omega$, $L = 50\ \text{mH}$, $V_m = 10\ \text{V}$ and $f = 50\ \text{Hz}$, so $\omega L = 15.71\ \Omega$ and

$$I_m = \frac{10}{\sqrt{20^2 + 15.71^2}} = 0.393\ \text{A}, \quad \phi = -38.1^\circ$$

$i_{ss}(0) = 0.309\ \text{A}$, so with $i(0) = 0$, $K = -0.309\ \text{A}$. Since $\tau = L/R = 2.5\ \text{ms}$, the transient dies out quickly compared with the 20 ms period.

3. RLC sinusoidal forcing: transient plus steady state

A stable RLC circuit driven sinusoidally responds with its natural modes plus a sinusoid at the forcing frequency; a linear time-invariant circuit never changes the forcing frequency.

Complete response

For an underdamped circuit, $x = x_n + x_{ss}$:

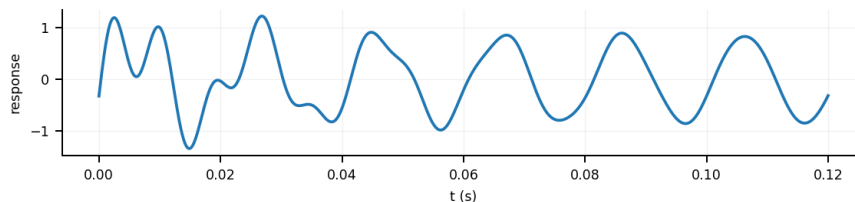
$$x(t) = e^{-\alpha t} [A \cos \omega_d t + B \sin \omega_d t] + X_m \cos(\omega t + \phi)$$

Steady-state amplitude and phase

X_m and ϕ come from $H(j\omega)$. The natural frequency ω_d belongs to the circuit poles; the forcing frequency ω belongs to the source. In general they differ.

Initial conditions select the transient

At switch-on the natural term is not optional: its two constants make the total state and its derivative match the stored energy at $t = 0$.



Worked example — identify the two frequencies

An RLC system with poles $-40 \pm j600\ \text{s}^{-1}$ is driven at $\omega = 200\ \text{rad/s}$. The transient rings at $600\ \text{rad/s}$ inside an e^{-40t} envelope, while the steady-state term oscillates at exactly $200\ \text{rad/s}$. The first few cycles can therefore look like beating even though the system is linear: during the transient the output is simply the sum of two frequencies.

4. Series resonance, natural frequency and damping

The sinusoidal impedance of a series RLC is given below. Resonance occurs when the reactive terms cancel: the impedance is purely resistive and, for a fixed source amplitude, the current is maximal.

$$Z(j\omega) = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Resonant frequency

For ideal L and C , $\omega_0 = 1/\sqrt{LC}$. At $\omega = \omega_0$, $\omega L = 1/(\omega C)$, so $Z = R$.

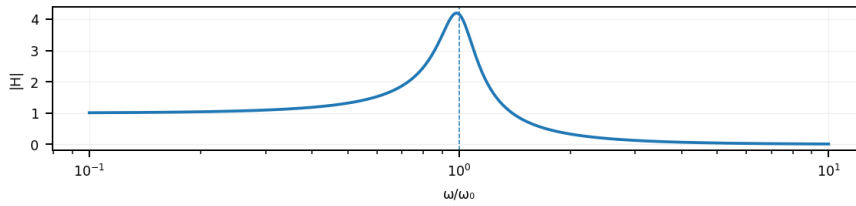
Quality factor

Higher Q means lower damping and a narrower, taller current resonance:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{2\zeta} \quad (\text{series RLC})$$

Connection to the transient poles

The same R , L and C also define $\alpha = R/(2L)$ and $\zeta = \alpha/\omega_0$. Frequency-domain selectivity and time-domain ringing are two views of the same pole geometry.



Worked example — resonant current and Q

$R = 10\ \Omega$, $L = 10\ \text{mH}$ and $C = 10\ \mu\text{F}$:

$$\omega_0 = \frac{1}{\sqrt{10^{-2} \cdot 10^{-5}}} = 3162\ \text{rad/s} \quad (f_0 = 503\ \text{Hz}), \quad Q = \frac{\omega_0 L}{R} = 3.162$$

With a 2 V peak source at resonance, the peak current is simply $2/10 = 0.2\ \text{A}$.

5. Off-resonance forcing and phase behaviour

Away from resonance, the net reactance changes both the current magnitude and its phase. Below resonance a series RLC is net capacitive; above resonance it is net inductive.

$$|I| = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \quad \phi_Z = \tan^{-1} \left[\frac{\omega L - 1/(\omega C)}{R} \right], \quad \phi_I = -\phi_Z$$

Below resonance

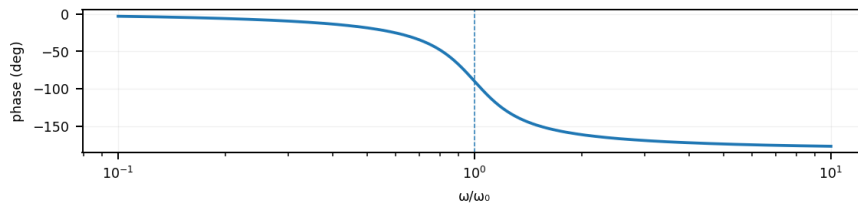
If $\omega L < 1/(\omega C)$, the reactance $X = \omega L - 1/(\omega C)$ is negative, the impedance angle is negative, and the current leads the source voltage.

At resonance

$X = 0$ and the current is in phase with the source; in the ideal series model the resistor receives the full source voltage.

Above resonance

$X > 0$ and the current lags the source voltage. Far above resonance the inductor dominates and the current magnitude falls roughly as $1/(\omega L)$.



Worked example — lead or lag?

For $R = 20 \, \Omega$, $L = 20 \, \text{mH}$ and $C = 20 \, \mu\text{F}$ at $\omega = 1000 \, \text{rad/s}$: $\omega L = 20 \, \Omega$ and $1/(\omega C) = 50 \, \Omega$, so $X = -30 \, \Omega$ and

$$\phi_Z = \tan^{-1}\left(\frac{-30}{20}\right) = -56.3^\circ$$

The current therefore leads the source voltage by 56.3° .

6. Initial phase and non-zero stored energy

A sinusoidal source can start at any phase, and the circuit may already hold stored energy. Both affect only the transient coefficients and the phase of the forced term; the pole locations do not change.

General forcing

For $v_s = V_m \cos(\omega t + \theta) u(t)$, the steady-state response has phase $\theta + \arg H(j\omega)$:

$$x_{ss} = X_m \cos(\omega t + \theta + \phi)$$

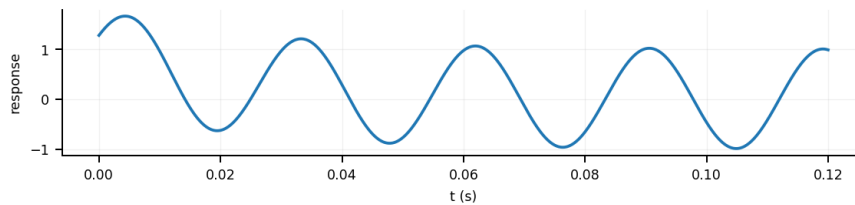
Non-zero state

The capacitor voltage and inductor current at 0^- supply the two physical initial conditions. Evaluate the forced solution and its derivative at 0, then choose the natural coefficients so that the sums match:

$$x(0) = x_n(0) + x_{ss}(0), \quad x'(0) = x_n'(0) + x_{ss}'(0)$$

Do not force the transient to zero

The transient vanishes only if the initial state happens to lie exactly on the steady-state trajectory at the switching phase. Otherwise the natural modes are required.



Worked example — natural coefficients in general

Let $x_n = e^{-\alpha t}[A\cos\omega_d t + B\sin\omega_d t]$ and $x_{ss} = X_m\cos(\omega t + \psi)$. At $t = 0$, $A = x(0) - X_m\cos\psi$. Differentiating, $x_n'(0) = -\alpha A + \omega_d B$ and $x_{ss}'(0) = -\omega X_m\sin\psi$, hence

$$B = \frac{x'(0) + \omega X_m\sin\psi + \alpha A}{\omega_d}$$

7. Laplace complete response versus phasor steady state

Laplace and phasor methods answer different questions. Laplace analysis keeps switching and stored energy; phasor analysis deliberately discards transients and describes one sinusoidal steady state.

Laplace method

Use causal inputs, initial conditions and rational functions of s . The inverse transform gives both the natural and forced components: complete response = transient + steady state.

Phasor method

Assume sinusoidal steady state from the start and replace d/dt by $j\omega$. The result is only the sinusoidal component left after all stable transients have decayed.

Consistency check

For a stable LTI circuit, the sinusoidal term from Laplace analysis must equal the phasor result at the same frequency. A discrepancy means an algebra, amplitude-convention or phase-reference error.

$$H(j\omega) = H(s)|_{s=j\omega}$$

Laplace

transient + steady state
initial conditions retained

Phasor

sinusoidal steady state only
transients assumed decayed

same steady-state sinusoid when assumptions agree

Worked example — RC low-pass

Laplace analysis of a switched-on sinusoid gives $v_C(t) = Ke^{-t/RC} + V_m|H(j\omega)|\cos(\omega t + \phi)$; phasor analysis gives only $V_m|H|\angle\phi$. After several time constants $Ke^{-t/RC} \approx 0$, and the Laplace solution reduces to the phasor result. The methods are consistent, not competing.

8. Periodic square-wave forcing using a step-series model

A periodic square wave is an infinite sequence of alternating step changes. For a stable LTI circuit, the response is the superposition of correspondingly shifted step responses.

Step-series representation

For a 0-to- V square wave of period T (half-period $T/2$):

$$v_i = V \sum_{k=0}^{\infty} \left[u(t - kT) - u\left(t - kT - \frac{T}{2}\right) \right]$$

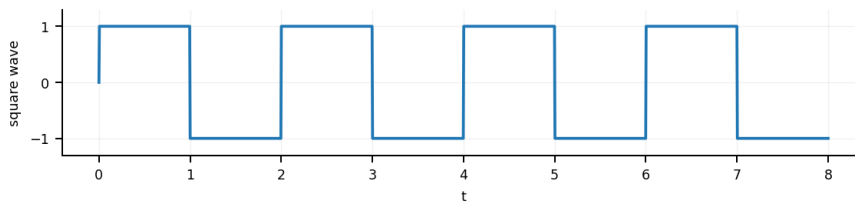
Transient accumulation

Each edge launches a new copy of the step response g . If the time constant is comparable with the half-period, old transients have not fully died away before the next edge:

$$y = V \sum_{k=0}^{\infty} \left[g(t - kT) - g\left(t - kT - \frac{T}{2}\right) \right]$$

Alternative Fourier viewpoint

A square wave can also be decomposed into odd harmonics and analysed with phasors term by term. Step-series analysis emphasises switching; Fourier analysis emphasises the periodic steady state.



Worked example — RC smoothing intuition

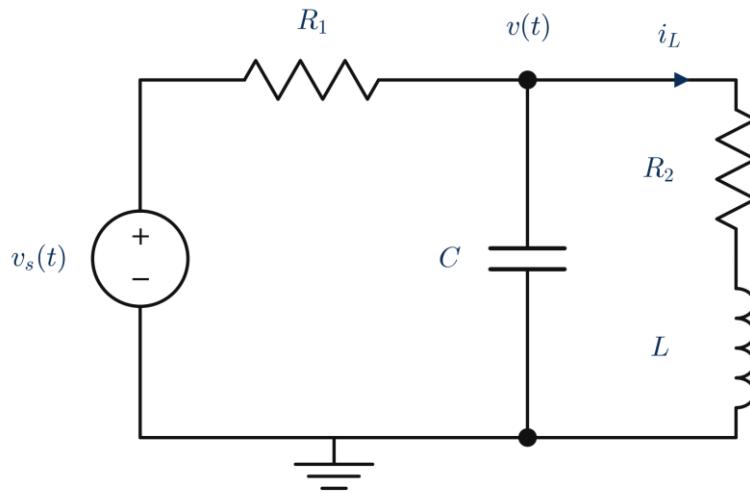
If $\tau \ll T/2$, the capacitor nearly reaches each new level before the next edge, and the output is a rounded square wave. If $\tau \gg T/2$, it changes only slightly each half-cycle and strongly smooths the wave. The step-series model predicts the transient after every edge; the Fourier model explains the same smoothing as attenuation of the high harmonics.

9. Capstone network: derive, transform, solve and interpret

A capstone problem demonstrates the full workflow: choose a state, derive the equation from KCL/KVL, transform with initial conditions, solve for the requested output, invert, and test the result physically.

Representative network

A source feeds R_1 into a node $v(t)$, with C to ground and a series R_2 - L branch from the same node to ground. The two storage elements make the network second order.



Capstone network: R_1 feeding C in parallel with a series R_2 - L branch.

Derive coupled first-order equations

KCL at the node and KVL around the R_2 - L branch give two equations that can be transformed directly or combined into one second-order ODE:

$$\frac{v_s - v}{R_1} = C v' + i_L, \quad v = R_2 i_L + L i_L'$$

Transform-domain solution

With zero initial state, eliminating $I_L(s)$ gives a second-order transfer function. Its poles fix the transient modes; evaluating it at $s = j\omega$ gives the sinusoidal steady state.

$$H(s) = \frac{V}{V_s} = \frac{R_2 + sL}{R_1 LC s^2 + (L + R_1 R_2 C)s + (R_1 + R_2)}$$

Worked example — numerical pole calculation

Let $R_1 = 10 \, \Omega$, $R_2 = 20 \, \Omega$, $L = 0.10 \, \text{H}$ and $C = 1 \, \text{mF}$. The denominator is $0.001s^2 + 0.30s + 30$, i.e. $s^2 + 300s + 30\,000$:

$$s = \frac{-300 \pm \sqrt{90\,000 - 120\,000}}{2} = -150 \pm j86.6 \, \text{s}^{-1}$$

The network is stable and underdamped. A complete sinusoidal switch-on response contains an e^{-150t} oscillation at 86.6 rad/s plus the forced sinusoid.

10. Course synthesis: modelling, method choice and verification

The central skill of this course is not memorising isolated formulas. It is moving reliably from a physical circuit to a mathematical model, choosing an efficient solution method, and checking that the answer is physically credible.



Model first

Define variables and references, apply KCL/KVL, insert the element laws, and identify the independent storage states. The ODE order comes from the number of independent storage states, not from appearance.

Choose the solution representation

Use direct ODE methods for simple forcing, Laplace methods for switching and initial conditions, phasors for sinusoidal steady state, and convolution when an impulse response is available.

Verify aggressively

Check units, state continuity (v_C and i_L under finite switching), initial derivative, DC final value, pole locations (stable poles lie in the left half-plane), limiting cases and energy. A correct-looking expression is not enough if it violates the circuit physics.

Final exam checklist

1. Mark reference directions and identify the state variables.
2. Write KCL/KVL before substituting memorised formulas.
3. Record the 0^- , 0^+ and ∞ values explicitly.
4. Classify the poles and time constants before fitting constants.
5. Use the supplied Laplace table strategically rather than re-deriving standard pairs.
6. Finish with at least two independent physical or algebraic checks.