

Worked Circuits III: Ramps, Pulses and Exponentials

Ordinary Differential Equations for Electrical Engineers — Ch. 14 of 15

1. RC response to a ramp input

A ramp exposes the difference between following a slowly changing input and matching it instantly. A first-order RC low-pass follows a ramp with a constant asymptotic lag.

Model

For the input $v_i = At u(t)$, the circuit obeys the equation below. With zero initial voltage, an integrating factor or a Laplace transform gives the response.

$$RC v_o' + v_o = At$$

Response

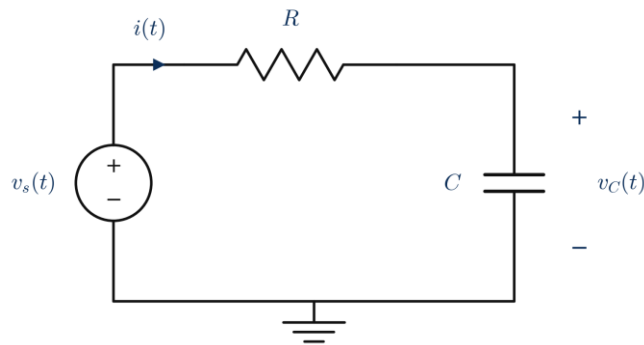
With $\tau = RC$, the output starts with zero slope, because the capacitor voltage cannot jump onto the ramp. At late times it has the same slope A but lags by $A\tau$.

$$v_o(t) = A[t - \tau + \tau e^{-t/\tau}]$$

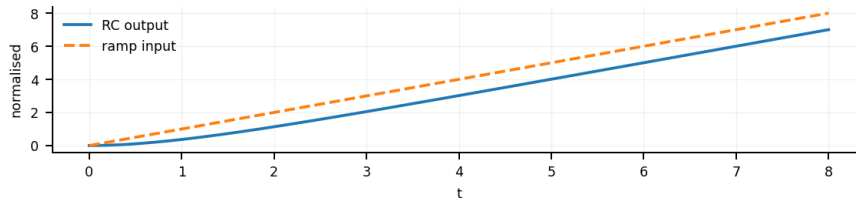
Tracking error

The error tends to $A\tau$: a smaller time constant improves ramp tracking, at the cost of a wider bandwidth.

$$e(t) = v_i - v_o = A\tau(1 - e^{-t/\tau})$$



RC low-pass: input $v_s(t)$, output $v_C(t)$.



Worked example — a 2 V/s ramp into an RC with $\tau = 0.5$ s

With $A = 2$ V/s and $\tau = 0.5$ s:

$$v_o = 2[t - 0.5 + 0.5 e^{-2t}] \text{ V}$$

At $t = 2$ s, $v_o = 3.018$ V while the input is 4 V. The asymptotic lag is $A\tau = 1$ V.

2. RC response to a finite rectangular pulse

A rectangular pulse is the difference of two delayed steps, which gives the whole charge-and-discharge waveform in one expression.

Input decomposition

A pulse of amplitude V from 0 to T is

$$v_i = V[u(t) - u(t - T)]$$

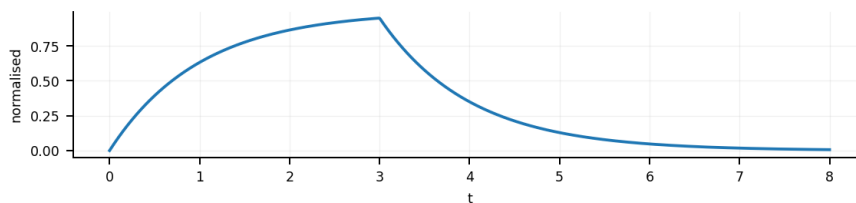
Use the step response

With the unit-step response $g(t) = 1 - e^{-t/\tau}$, the output is $v_o = Vg(t) - Vg(t - T)u(t - T)$:

$$v_o = V[(1 - e^{-t/\tau}) - (1 - e^{-(t-T)/\tau})u(t - T)]$$

Continuity at the falling edge

At $t = T$ the forcing changes instantly, so the derivative changes, but the capacitor voltage itself does not jump: $v_o(T^+) = v_o(T^-)$.



Worked example — pulse of width 2τ

At $t = T = 2\tau$ the capacitor has reached $V(1 - e^{-2}) = 0.865V$. After the pulse ends it discharges from that value:

$$v_o(t) = 0.865 V e^{-(t-T)/\tau}, \quad t \geq T$$

The step-difference expression gives exactly the same result, which is a useful exam check.

3. RL response to a pulse: current continuity at both edges

An RL pulse problem has two switching instants, and the inductor current must be continuous at both. The current builds during the pulse and decays after the source is removed.

During the pulse

For $0 < t < T$ with zero initial current and $\tau = L/R$:

$$i(t) = \frac{V}{R}(1 - e^{-t/\tau}), \quad i(T^-) = \frac{V}{R}(1 - e^{-T/\tau})$$

At pulse removal

$i(T^+) = i(T^-)$: the source disappears but the inductor current cannot. The post-pulse response is source-free:

$$i(t) = i(T) e^{-(t-T)/\tau}, \quad t \geq T$$

Step-superposition form

The whole current can also be written as the difference of two delayed step responses, often the faster route in Laplace problems:

$$i = \frac{V}{R}[g(t) - g(t - T)u(t - T)]$$

Worked example — pulse width equal to one time constant

$V = 12 \text{ V}$, $R = 6 \Omega$ and $L = 60 \text{ mH}$, so $\tau = 10 \text{ ms}$ and $I_\infty = 2 \text{ A}$. For $T = \tau$, $i(T) = 2(1 - e^{-1}) = 1.264 \text{ A}$. Just after switch-off the current is still 1.264 A ; ten milliseconds later it is $1.264/e = 0.465 \text{ A}$.

4. First-order response to an exponential input

An exponential input can interact strongly with the circuit's natural exponential. If the input exponent matches the natural pole, the usual particular form must be multiplied by t .

General forcing

For the equation below with $b \neq -a$, try $y_p = Ae^{bt}$; substitution gives $A = K/(a + b)$.

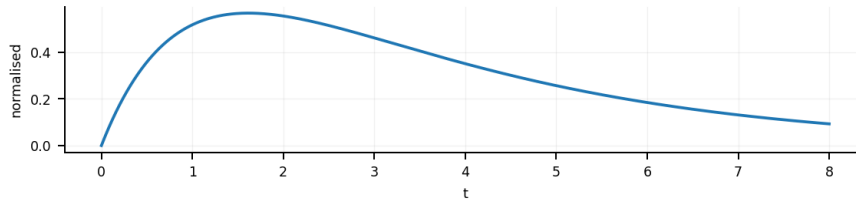
$$y' + ay = Ke^{bt}, \quad y_p = \frac{Ke^{bt}}{a + b}$$

The resonant algebraic case

If $b = -a$, the trial duplicates the homogeneous solution, so use $y_p = Kte^{-at}$ instead. This is the first-order version of repeated forcing roots.

Circuit interpretation

Exponential test inputs are useful because LTI systems map exponentials to scaled exponentials, except when the input exponent coincides with a system pole.



Worked example — RC driven by a decaying exponential

Solve $y' + y = 2e^{-0.35t}$ with $y(0) = 0$. Here $A = 2/(1 - 0.35) = 3.0769$, so $y = 3.0769 e^{-0.35t} + C e^{-t}$, and $y(0) = 0$ gives $C = -3.0769$:

$$y = 3.0769 (e^{-0.35t} - e^{-t})$$

5. RLC response to a finite pulse using shifted steps

A second-order circuit responds to a pulse as the difference of two complete step responses, so the RLC initial-value problem need not be re-solved after the falling edge.

Pulse representation

For $v_i = V[u(t) - u(t - T)]$ and an LTI circuit with unit-step response $g(t)$, the zero-state response is

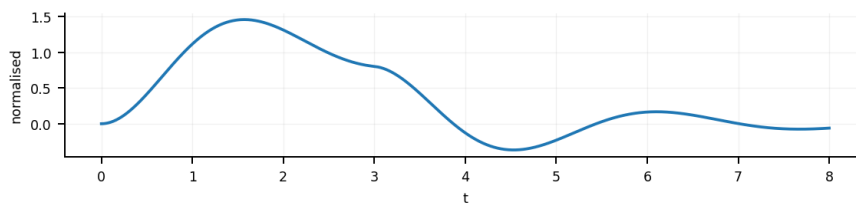
$$y_{zs} = V[g(t) - g(t - T) u(t - T)]$$

The delayed copy carries the correct states

The second term is not a naive reset-to-zero response: superposing the two zero-state step responses automatically reproduces the correct continuity of capacitor voltage and inductor current.

Take care with non-zero initial energy

If energy is already stored at $t = 0$, add the zero-input response separately: $y = y_{zi} + y_{zs}$. The shifted-step construction gives only the zero-state part.



Worked example — structure of an underdamped pulse response

Suppose $g(t) = 1 - e^{-\alpha t} [\cos \omega_d t + (\alpha/\omega_d) \sin \omega_d t]$. For a 4 V pulse of width T :

$$y(t) = 4g(t) - 4g(t - T) u(t - T)$$

After $t = T$ the two ringing waveforms interfere; depending on T they can reinforce or partly cancel.

6. Piecewise excitation and re-initialisation at each interval

Interval-by-interval solution is still essential when the topology changes, not just a source value. Each switching instant starts a new initial-value problem that inherits its state from the previous interval.

Solve interval by interval

On each interval, write the ODE for the active topology and find the interval's final value and time constant (or pole structure).

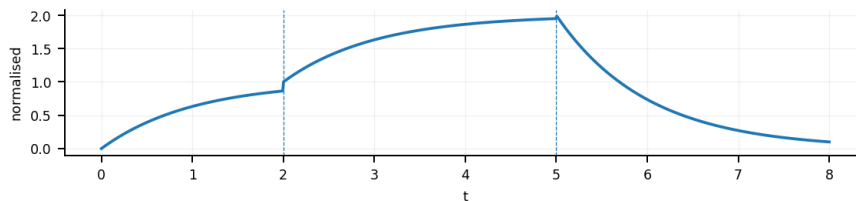
Carry state continuity across the boundary

At $t = T$, set $v_c(T^+) = v_c(T^-)$ and $i_L(T^+) = i_L(T^-)$ unless an impulse explicitly changes the state. These become the initial conditions of the next interval.

Reset local time

Using local time $\theta = t - T$ on the new interval keeps formulas clear:

$$x(t) = x_{\infty,2} + [x(T) - x_{\infty,2}]e^{-(t-T)/\tau_2}$$



Worked example — three RC intervals

1. $0 \leq t < 2\tau$: charge from 0 towards 1 V.
2. $2\tau \leq t < 5\tau$: the source switches so the new final value is 2 V; start from $x(2\tau) = 1 - e^{-2}$.
3. $t \geq 5\tau$: source removed; discharge from $x(5\tau)$ towards 0.

Writing each interval in local time makes continuity visible and avoids hidden sign errors.

7. Superposition for sums of delayed inputs

Linear circuits let complicated switching waveforms be split into simple delayed components; the total response is the sum of the component responses.

Input decomposition

Represent each change of level as $\Delta V_k u(t - T_k)$. For a zero-state system with step response g :

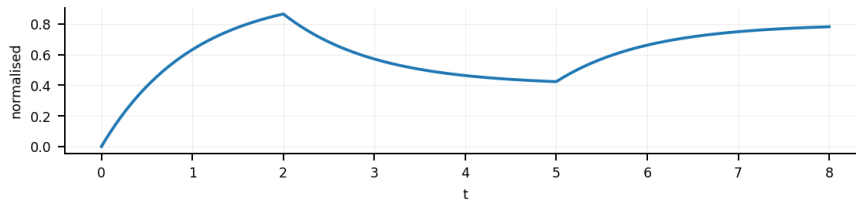
$$v_i(t) = \sum_k \Delta V_k u(t - T_k) \quad \Rightarrow \quad y(t) = \sum_k \Delta V_k g(t - T_k) u(t - T_k)$$

Why level changes beat absolute levels

Each ΔV_k is the change at a switching edge, which avoids double-counting the baseline and gives a compact expression.

Limits of superposition

The method requires linearity: the response to a sum is the sum of the responses. Nonlinear resistors, saturation, and switching devices operating outside an ideal linear model invalidate it.



Worked example — a three-level source

The input changes by +5 V at $t = 0$, -3 V at $t = 2\tau$ and +2 V at $t = 5\tau$. With $g(t) = 1 - e^{-t/\tau}$:

$$v_o = 5g(t) - 3g(t - 2\tau)u(t - 2\tau) + 2g(t - 5\tau)u(t - 5\tau)$$

No new differential equation is solved at the later switching times.

8. Convolution using an RC impulse response

Convolution is a third route, besides direct ODE solution and Laplace algebra, to a circuit response. It is especially valuable for understanding arbitrary input waveforms.

Impulse response of the RC low-pass

$$H(s) = \frac{1}{1 + s\tau} \Rightarrow h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

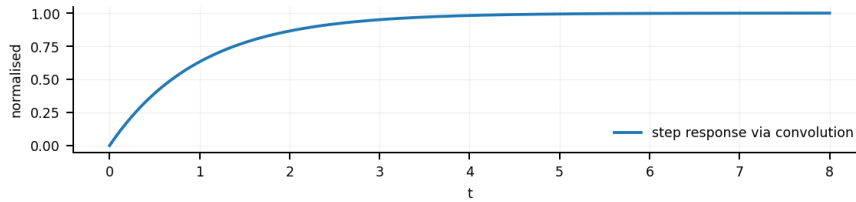
Convolution with an input

The kernel weights recent input values more heavily than distant ones, because of the exponential memory decay:

$$v_o(t) = \int_0^t h(t - \lambda) v_i(\lambda) d\lambda$$

The step as a special case

For $v_i = V u(t)$ the integral gives $V(1 - e^{-t/\tau})$: the familiar first-order response is an accumulated impulse response.



Worked example — constant input via convolution

With $\tau = 0.2$ s and $V = 3$ V:

$$v_o(t) = \int_0^t 5 e^{-5(t-\lambda)} \cdot 3 d\lambda = 15 e^{-5t} \int_0^t e^{5\lambda} d\lambda = 3(1 - e^{-5t}) \text{ V}$$

9. Initial- and final-value checks on worked circuits

The initial- and final-value theorems are fast consistency checks for rational Laplace solutions. Verify their assumptions first; they do not replace understanding the circuit.

Initial value

For an ordinary causal function with no hidden impulse at the origin, the result below must agree with capacitor-voltage or inductor-current continuity.

$$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$$

Final value

If all poles of $sF(s)$ lie strictly in the left half-plane (apart from the cancelled origin factor the theorem allows), then

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

When the final value fails

Sinusoidal poles on the imaginary axis or unstable right-half-plane poles mean no finite final value exists. Check the poles of $sF(s)$ before applying the theorem.

Initial-value theorem

$$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$$

Final-value theorem

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

valid only when all poles of $sF(s)$ lie in the left half-plane

Worked example — check an RC pulse transform

For an RC pulse, $V_o(s) = \frac{V(1-e^{-sT})}{s(1+s\tau)}$.

$$\lim_{s \rightarrow \infty} sV_o = 0, \quad \lim_{s \rightarrow 0} sV_o = \lim_{s \rightarrow 0} \frac{V(1 - e^{-sT})}{1 + s\tau} = 0$$

Both match the physical waveform: the capacitor starts and eventually ends at 0 V.

10. Comparing responses and common modelling mistakes

Ramp, pulse, exponential and piecewise inputs all use the same circuit dynamics. The challenge is to represent the forcing correctly and carry the state through switching events.

Common modelling mistakes

Forgetting the unit step on a delayed waveform (a delay always needs $u(t - T)$), resetting a capacitor voltage or inductor current to zero at a switch, and using the wrong post-switch topology are more serious than minor algebra slips.

Choose the most economical representation

Use shifted steps for level changes, interval-by-interval analysis for topology changes, Laplace transforms for many delays, and convolution when the impulse response is already known.

Verification strategy

Check causality (a delayed input gives a delayed response), continuity, dimensions, initial slope, final value, and limiting cases such as $T \rightarrow 0$ or $T \rightarrow \infty$ for pulse problems.

Checkpoint — choose the right tool

1. Finite pulse, unchanged topology: shifted-step superposition is usually shortest.
2. Switch changes R or the connections: interval-by-interval re-initialisation is often clearest.
3. Arbitrary measured waveform with known $h(t)$: convolution is natural.
4. Several delays with a rational system model: Laplace algebra is often most efficient.