

Worked Circuits II: Second-Order RLC Transients

Ordinary Differential Equations for Electrical Engineers — Ch. 13 of 15

1. Series-RLC source-free response from stored energy

With the source removed, an RLC circuit still responds: energy stored in L or C at the switching instant drives the natural modes. Two independent initial conditions fix the modal coefficients.

Model

For the series-RLC capacitor voltage, the initial data are $v_C(0^+)$ and $v_C'(0^+) = i(0^+)/C$:

$$LC v_C'' + RC v_C' + v_C = 0$$

Classify before solving

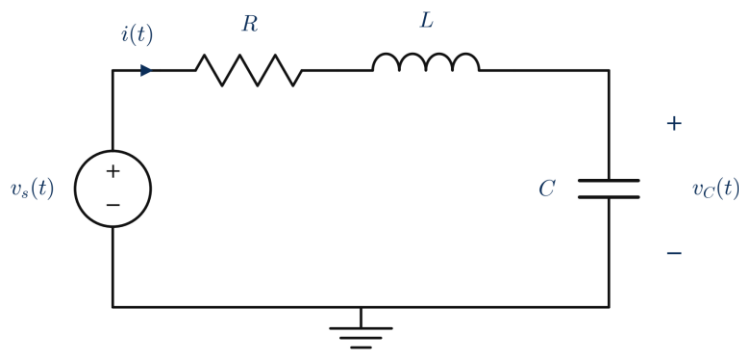
The ratio of α to ω_0 decides whether the natural response is over-, critically or underdamped. Classify before fitting any constants.

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Stored energy sets the amplitude

Resistance removes energy, so a passive source-free response must decay towards zero from the initial energy

$$W(0) = \frac{1}{2} C v_C^2(0) + \frac{1}{2} L i^2(0)$$



Series RLC circuit used in Sections 1–4.

Worked example — initial capacitor voltage only

$R = 10 \Omega$, $L = 50 \text{ mH}$, $C = 100 \mu\text{F}$, $v_C(0) = 5 \text{ V}$ and $i(0) = 0$. Then $\alpha = 100 \text{ s}^{-1}$, $\omega_0 = 447.2 \text{ rad/s}$ and $\omega_d = 435.9 \text{ rad/s}$, so

$$v_C(t) = e^{-100t} [A \cos 435.9t + B \sin 435.9t]$$

$A = 5$, and $v_C'(0) = 0$ gives $-100A + 435.9B = 0$, so $B = 1.147$.

2. Series-RLC DC-step response: overdamped case

When $\alpha > \omega_0$, a DC step gives a non-oscillatory response: two real exponential modes plus the final DC value.

Roots

Both natural poles are negative for a passive stable circuit:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

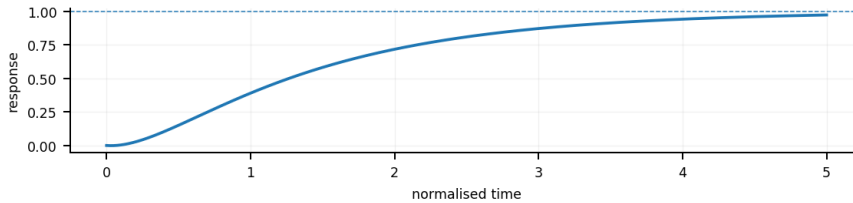
Initial conditions

For a step from an unenergised state, $v_C(0) = 0$ and $i(0) = C v_C'(0) = 0$. These two conditions fix C_1 and C_2 :

$$v_C = V_s + C_1 e^{s_1 t} + C_2 e^{s_2 t}, \quad C_1 + C_2 = -V_s, \quad s_1 C_1 + s_2 C_2 = 0$$

Dominant pole

At late times the pole nearer the origin dominates, so an overdamped response can look approximately first order once the faster mode has decayed.



Worked example — compute the two exponential coefficients

$V_s = 10 \text{ V}$, $L = 0.10 \text{ H}$, $C = 100 \text{ } \mu\text{F}$ and $R = 100 \text{ } \Omega$, so $\alpha = 500 \text{ s}^{-1}$ and $\omega_0 = 316.2 \text{ rad/s}$, giving $s_1 = -112.7 \text{ s}^{-1}$ and $s_2 = -887.3 \text{ s}^{-1}$. Solving the two conditions gives $C_1 = -11.46$ and $C_2 = 1.46$:

$$v_C = 10 - 11.46 e^{-112.7t} + 1.46 e^{-887.3t} \text{ V}$$

3. Series-RLC DC-step response: critically damped case

Critical damping occurs at $\alpha = \omega_0$. The poles merge, so the natural response needs the independent functions $e^{-\alpha t}$ and $t e^{-\alpha t}$.

Response form

Two constants remain, because the circuit is still second order even though the pole is repeated:

$$v_C = V_s + (C_1 + C_2 t) e^{-\alpha t}$$

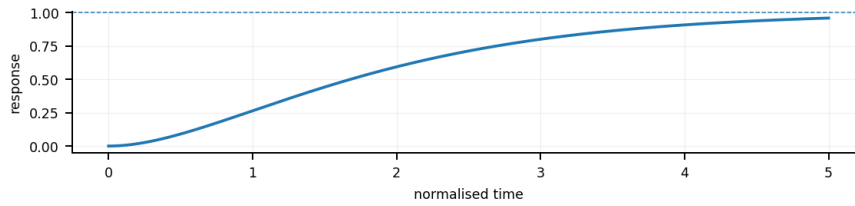
Initial rest conditions

If $v_C(0) = 0$ and $i(0) = 0$, then $C_1 = -V_s$, and differentiating at $t = 0$ gives $C_2 = -\alpha V_s$:

$$v_C = V_s [1 - (1 + \alpha t) e^{-\alpha t}], \quad R_{\text{crit}} = 2 \sqrt{\frac{L}{C}} \text{ (series RLC)}$$

Design interpretation

Critical damping is the boundary between monotonic (overdamped) and oscillatory (underdamped) behaviour. Because of component tolerances, it is an ideal design point rather than a precisely achievable condition.



Worked example — critically damped 5 V step

With $L = 40 \text{ mH}$ and $C = 10 \text{ }\mu\text{F}$, $R_{\text{crit}} = 126.5 \Omega$ and $\alpha = \omega_0 = 1581 \text{ s}^{-1}$. For a 5 V step from zero:

$$v_C(t) = 5[1 - (1 + 1581t) e^{-1581t}] \text{ V}$$

Both $v_C(0) = 0$ and $v_C'(0) = 0$ are satisfied.

4. Series-RLC DC-step response: underdamped case

For $\alpha < \omega_0$ the step response rings about its final value, at frequency ω_d inside an envelope decaying at rate α .

Standard form

With zero initial state, this is the canonical underdamped second-order low-pass step response:

$$v_C = V_s \left\{ 1 - e^{-\alpha t} \left[\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t \right] \right\}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

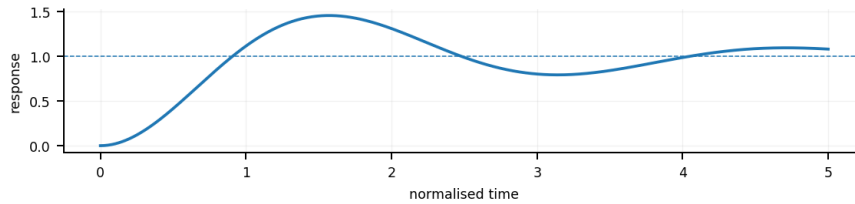
Overshoot

The first peak occurs when the derivative first returns to zero after $t = 0$. For the standard normalised form, the fractional overshoot depends only on $\zeta = \alpha/\omega_0$:

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

Settling

A common estimate of the 2% settling time is $4/\alpha$ when ζ is not very small. It comes from the exponential envelope, not the oscillation period.



Worked example — damping ratio and overshoot

Take $\zeta = 0.30$ and $\omega_0 = 1000 \text{ rad/s}$. Then $\alpha = 300 \text{ s}^{-1}$ and $\omega_d = 953.9 \text{ rad/s}$.

$$M_p = \exp \left[\frac{-\pi(0.30)}{\sqrt{1-0.09}} \right] = 0.372 \quad (37.2\%)$$

The 2% settling time is about $4/300 = 13.3 \text{ ms}$, and the ringing period is $2\pi/953.9 = 6.59 \text{ ms}$.

5. Parallel-RLC natural response using node voltage

A parallel RLC has the same ideal natural frequency $1/\sqrt{LC}$, but its damping depends on the conductance $1/R$ rather than a series resistance.

Node-voltage ODE

With no source, KCL gives $Cv' + v/R + i_L = 0$. Differentiating and using $i_L' = v/L$:

$$v'' + \frac{1}{RC} v' + \frac{1}{LC} v = 0$$

Damping factor

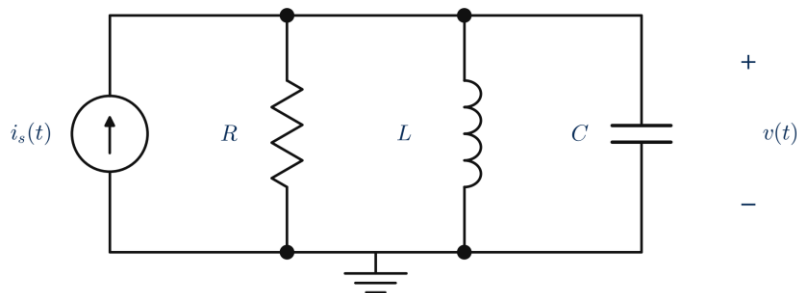
Comparing with $v'' + 2\alpha v' + \omega_0^2 v = 0$ gives the result below. A smaller shunt resistance means more damping; a larger one approaches an ideal LC tank.

$$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

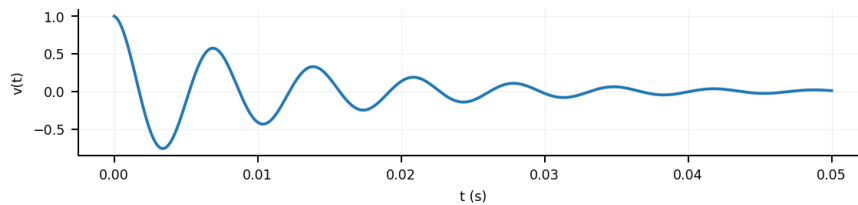
Initial conditions

The natural state is still set by $v_c(0)$ and $i_L(0)$, and KCL at 0^+ gives the initial slope:

$$v'(0^+) = \frac{-v(0^+)/R - i_L(0^+)}{C}$$



Parallel RLC tank (source removed for the natural response).



Worked example — classify a parallel tank

$R = 500 \Omega$, $L = 20 \text{ mH}$ and $C = 5 \mu\text{F}$ give $\omega_0 = 3162 \text{ rad/s}$ and $\alpha = 1/(2RC) = 200 \text{ s}^{-1}$, so $\zeta = 0.063$. The circuit is lightly damped and rings strongly after an initial-energy disturbance; $\omega_d \approx 3156 \text{ rad/s}$, only slightly below ω_0 .

6. Initial derivatives from KCL/KVL

Problems usually give the capacitor voltage and inductor current as physical initial data, but the mathematical solution needs a state and its derivative. KCL or KVL at 0^+ supplies the missing derivative.

Series circuit

If v_C is the state, the first relation applies directly; if the current is chosen, use KVL at 0^+ :

$$v_C'(0^+) = \frac{i(0^+)}{C}, \quad i'(0^+) = \frac{v_s(0^+) - Ri(0^+) - v_C(0^+)}{L}$$

Parallel circuit

If the node voltage is chosen, KCL gives

$$v'(0^+) = \frac{i_s(0^+) - v(0^+)/R - i_L(0^+)}{C}$$

Why this beats differentiating guessed solutions

Physical initial conditions are tied to stored energy and circuit laws. Converting them before solving prevents sign mistakes and makes both required conditions explicit.



Worked example — series RLC initial slope

At $t = 0^+$: $v_s = 8 \text{ V}$, $R = 4 \Omega$, $L = 20 \text{ mH}$, $v_C = 3 \text{ V}$ and $i = 0.5 \text{ A}$.

$$i'(0^+) = \frac{8 - 4(0.5) - 3}{0.020} = 150 \text{ A/s}$$

If the variable is v_C instead, then with $C = 100 \mu\text{F}$, $v_C'(0^+) = i/C = 5000 \text{ V/s}$. These derivatives are not optional: they are the second initial condition in different coordinates.

7. Peak overshoot, ringing frequency and settling

Response specifications translate pole locations into measurable time-domain quantities. They are especially useful when RLC circuits serve as prototypes for general second-order systems.

Damping ratio

$\zeta = \alpha/\omega_0$ normalises the damping. For $0 < \zeta < 1$:

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

Peak time and overshoot

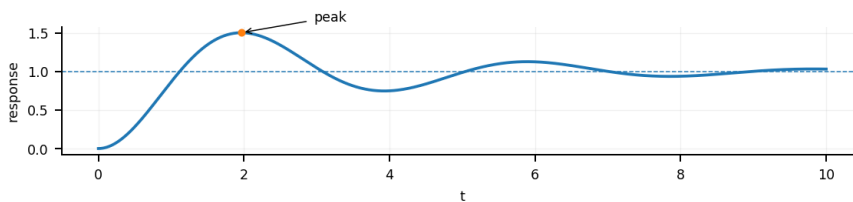
For the standard underdamped low-pass step response with zero initial state:

$$t_p = \frac{\pi}{\omega_d}, \quad M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

Settling time

The envelope is $e^{-\zeta\omega_0 t} = e^{-\alpha t}$, giving the common 2% estimate below. It is an engineering approximation, not an exact crossing time for every waveform.

$$t_s(2\%) \approx \frac{4}{\zeta\omega_0} = \frac{4}{\alpha}$$



Worked example — specifications from ζ and ω_0

With $\zeta = 0.4$ and $\omega_0 = 800$ rad/s: $\omega_d = 733.2$ rad/s, so $t_p = 4.29$ ms; $M_p = 25.4\%$; and $t_s \approx 4/(0.4 \times 800) = 12.5$ ms.

8. Energy exchange between L and C with resistor loss

In an underdamped RLC circuit, energy oscillates between the capacitor's electric field and the inductor's magnetic field, while resistance irreversibly removes total energy.

Instantaneous stored energy

In an ideal LC circuit the sum of the two terms is constant, and they trade energy periodically.

$$W_C = \frac{1}{2} C v_C^2, \quad W_L = \frac{1}{2} L i^2$$

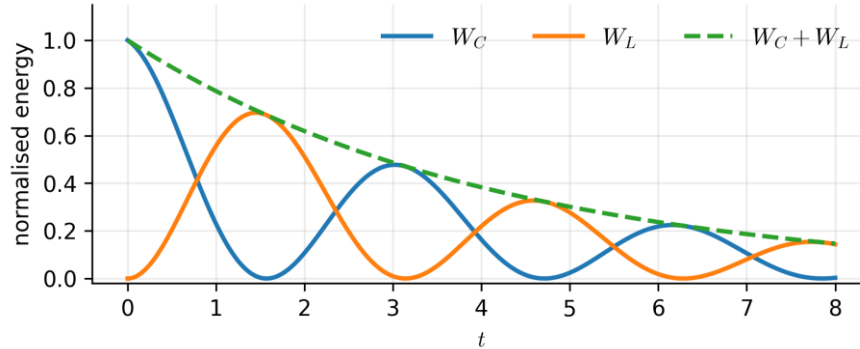
Resistive loss

The resistor dissipates $p_R = Ri^2 \geq 0$. For a source-free series RLC, the total stored energy decreases monotonically even though each storage term oscillates:

$$\frac{d}{dt}(W_C + W_L) = -Ri^2$$

Damping as energy decay

State amplitudes decay roughly as $e^{-\alpha t}$; energy, being quadratic in the state, decays on an $e^{-2\alpha t}$ envelope for a pure modal response.



Worked example — energy bookkeeping

A source-free series RLC has $C = 10 \mu\text{F}$, $L = 10 \text{ mH}$, $v_C(0) = 20 \text{ V}$ and $i(0) = 0$. The initial energy is $1/2 (10 \mu\text{F})(20^2) = 2.0 \text{ mJ}$, so for $R > 0$, $W_C + W_L \leq 2.0 \text{ mJ}$ at all later times. A numerical solution showing stored energy growing with no source means the model, the sign convention or the integration is wrong.

9. Laplace-domain verification of the same solution

A good habit is to solve or classify in the time domain, then independently verify the pole structure and initial-condition terms in the Laplace domain.

Denominator check

The transformed denominator must match the characteristic polynomial of the homogeneous ODE: ODE roots $\leftrightarrow$ denominator poles. If not, a coefficient or transform rule has been misapplied.

Initial-condition check

The numerator contains terms from the initial voltage and current as well as the input. Inversion must reproduce the same two modal coefficients found by applying the initial conditions directly.

Limit checks

Use the initial-value theorem and, when the pole conditions hold, the final-value theorem. They check the algebra without repeating the full inversion.

$$f(0^+) = \lim_{s \rightarrow \infty} sF(s), \quad f(\infty) = \lim_{s \rightarrow 0} sF(s)$$



Worked example — verify a known step response

Suppose

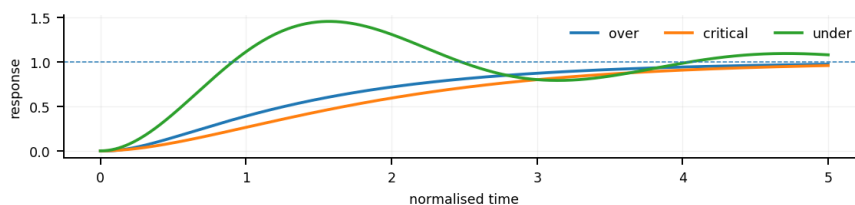
$$V_C(s) = \frac{10 \times 100\,000}{s(s^2 + 200s + 100\,000)}$$

The poles are $-100 \pm j300$, matching $\alpha = 100$ and $\omega_d = 300$. $\lim_{s \rightarrow \infty} sV_C(s) = 0$ confirms $v_C(0^+) = 0$, and $\lim_{s \rightarrow 0} sV_C(s) = 10$ confirms the DC final value.

10. Comparing the damping classes: degree-level checkpoint

The three damping classes are different pole geometries of the same second-order equation. Classify first, in exams and in design work, before any detailed algebra.

Class	Condition	Poles and behaviour
overdamped	$\alpha > \omega_0$	two distinct negative real poles; no ringing; one mode often dominates late time
critically damped	$\alpha = \omega_0$	repeated negative real pole; response includes $t e^{-\alpha t}$; boundary between non-oscillatory and oscillatory
underdamped	$\alpha < \omega_0$	complex-conjugate pair; ringing frequency from the imaginary part, envelope from the real part



Checkpoint — classify without solving

1. $x'' + 900x' + 160\,000x = 0$: $\alpha = 450$, $\omega_0 = 400 \Rightarrow$ overdamped.
2. $x'' + 800x' + 160\,000x = 0$: $\alpha = 400$, $\omega_0 = 400 \Rightarrow$ critically damped.
3. $x'' + 300x' + 160\,000x = 0$: $\alpha = 150$, $\omega_0 = 400 \Rightarrow$ underdamped.

Only after classifying should you choose the response form.