

Worked Circuits I: First-Order RC and RL Switching

Ordinary Differential Equations for Electrical Engineers — Ch. 12 of 15

1. Worked method: schematic to verified solution

A repeatable workflow prevents most switching errors. Keep modelling, state continuity, final-value reasoning and solution construction separate, rather than mixing them in one line of algebra.

Step 1: define the state and references

Choose the capacitor voltage or inductor current as the primary state. Mark voltage polarities and current directions, and write KCL or KVL before inserting any memorised response form.

Step 2: find the 0^+ and final values

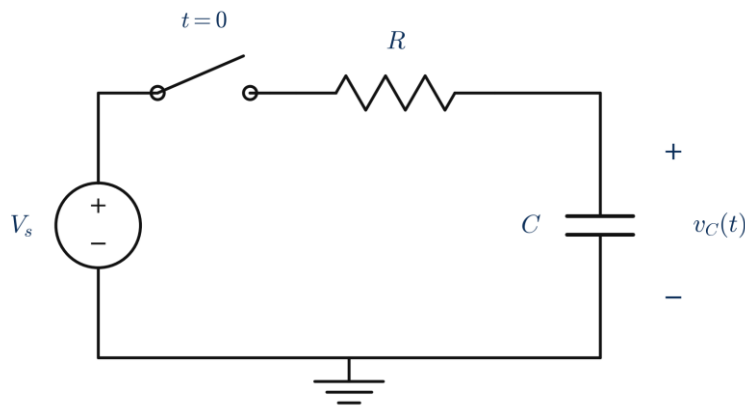
For finite switching, the state is continuous. Then, for the DC steady state (if one is reached), replace C by an open circuit and L by a short circuit.

$$v_C(0^+) = v_C(0^-), \quad i_L(0^+) = i_L(0^-)$$

Step 3: find τ and assemble

Find the resistance seen by the storage element with independent sources deactivated, giving the time constant below. Assemble the response and verify the initial slope against the original circuit law.

$$\tau_C = R_{th} C, \quad \tau_L = \frac{L}{R_{th}}, \quad x(t) = x_\infty + (x_0 - x_\infty)e^{-t/\tau}$$



A switched RC circuit: the capacitor voltage is the state carried across $t = 0$.

Mini-example — a 5 V RC step

$R = 10 \text{ k}\Omega$, $C = 10 \text{ }\mu\text{F}$ and $v_C(0^-) = 0$, so $\tau = 0.10 \text{ s}$ and $v_C(\infty) = 5 \text{ V}$:

$$v_C(t) = 5(1 - e^{-10t}) \text{ V}$$

Check: $v_C'(0^+) = 5/0.10 = 50 \text{ V/s}$, matching $(5 - 0)/(RC)$.

2. RC charging from a DC step with zero initial voltage

An initially uncharged capacitor connected through a resistor to a DC source is the standard first-order charging problem. The capacitor voltage rises continuously while the current decays exponentially.

Model

KVL gives $RC v_C' + v_C = V_s$. The capacitor starts at 0 V; after a long time it is an open circuit, the resistor drop is zero, and $v_C \rightarrow V_s$.

Response

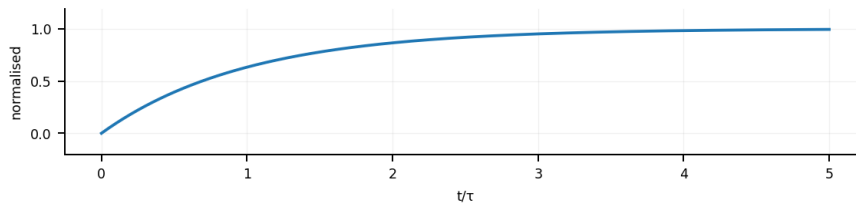
With $\tau = RC$, the voltage and current are given below. At $t = 0^+$ the capacitor holds its initial voltage (here 0 V), so the whole source voltage appears across R .

$$v_C(t) = V_s(1 - e^{-t/RC}), \quad i(t) = \frac{V_s}{R} e^{-t/RC}$$

Energy viewpoint

The source delivers energy to both C and R . Finally the capacitor stores the energy below; when charging from an ideal voltage source through a resistor, an equal amount is dissipated in R .

$$W_C(\infty) = \frac{1}{2} C V_s^2$$



Worked example — 12 V charging circuit

$R = 4.7 \text{ k}\Omega$, $C = 47 \text{ }\mu\text{F}$ and $V_s = 12 \text{ V}$, so $\tau = RC = 0.2209 \text{ s}$.

$$v_C(0.50) = 12[1 - e^{-0.50/0.2209}] = 10.75 \text{ V}$$

$i(0^+) = 12/4700 = 2.55 \text{ mA}$, while $i(0.50) = 0.265 \text{ mA}$.

3. RC discharge from a non-zero initial voltage

During discharge the source is removed and the stored capacitor energy drives current through the resistor. The final value is zero, but the state cannot jump at the switching instant.

Source-free ODE

KVL gives $RC v_C' + v_C = 0$, with natural response $V_0 e^{-t/RC}$. The current sign depends on the reference: with the passive convention (current entering the positive terminal), $i_C = C v_C'$ is negative during discharge.

$$v_C(t) = V_0 e^{-t/RC}, \quad i_C(t) = -\frac{V_0}{R} e^{-t/RC}$$

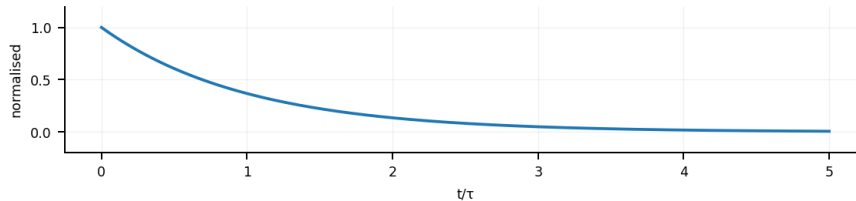
Time-constant meaning

After one time constant the voltage has fallen to $e^{-1} = 36.8\%$ of its initial value; after five it is below 1%.

Energy decays twice as fast in the exponent

Since $W_C = 1/2 C v_C^2$, the stored energy follows $e^{-2t/RC}$: voltage and energy do not share the same exponent.

$$W_C(t) = \frac{1}{2} C V_0^2 e^{-2t/RC}$$



Worked example — discharge to a safety threshold

A 100 μF capacitor at 24 V discharges through 12 k Ω , so $\tau = 1.2$ s. The time to reach 5 V follows from $5 = 24e^{-t/1.2}$:

$$t = 1.2 \ln \frac{24}{5} = 1.88 \text{ s}$$

The energy has then fallen to $(5/24)^2 = 4.34\%$ of its initial value.

4. RC switching between two DC levels

A capacitor need not start from zero. When a source changes from one DC level to another, the old steady-state voltage becomes the initial condition for the new interval.

Before switching

If the circuit has been in steady state for a long time, the capacitor voltage equals the pre-switch DC value V_1 . Record it as $v_C(0^-)$.

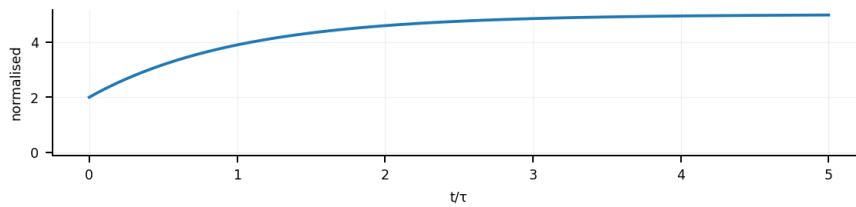
At the switching instant

For finite sources $v_C(0^+) = v_C(0^-) = V_1$. The new circuit fixes the final value V_2 and the time constant, and the response moves exponentially from the old level to the new.

General two-level formula

For a change from V_1 to V_2 through resistance R , the sign of the derivative is set automatically by $V_1 - V_2$:

$$v_C(t) = V_2 + (V_1 - V_2)e^{-t/RC}$$



Worked example — a 2 V to 8 V transition

$R = 3 \text{ k}\Omega$, $C = 100 \text{ }\mu\text{F}$, $V_1 = 2 \text{ V}$ and $V_2 = 8 \text{ V}$, so $\tau = 0.30 \text{ s}$ and

$$v_C(t) = 8 - 6e^{-t/0.30} \text{ V}$$

At $t = 0.30 \text{ s}$, $v_C = 8 - 6/e = 5.79 \text{ V}$: 63.2% of the total change is complete.

5. RL current build-up from a DC step

Inductor current is the magnetic analogue of capacitor voltage: it is continuous and approaches its new DC value exponentially after a step.

Model

KVL gives $L i' + Ri = V_s$. With $i(0^-) = 0$, the final current is V_s/R , because the inductor is a short circuit in DC steady state. The time constant is $\tau = L/R$.

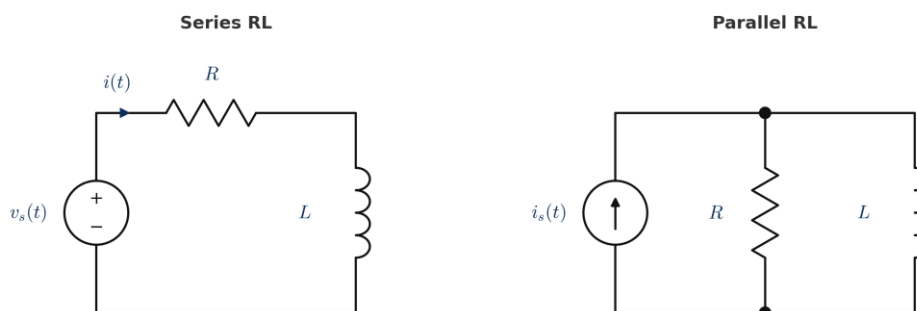
Current and inductor voltage

The inductor voltage starts at the full source voltage and decays to zero as the current becomes constant:

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau}), \quad v_L(t) = V_s e^{-t/\tau}$$

Energy

The inductor stores $1/2 Li^2$. During build-up the source supplies both the resistive loss and the growing magnetic energy.



The series RL circuit (left) is the model used here.

Worked example — current after 20 ms

$R = 8\ \Omega$, $L = 80\ \text{mH}$ and $V_s = 16\ \text{V}$, so $\tau = L/R = 10\ \text{ms}$ and $I_\infty = 2\ \text{A}$. At 20 ms:

$$i = 2(1 - e^{-2}) = 1.73\ \text{A}, \quad v_L = 16 e^{-2} = 2.17\ \text{V}$$

6. RL current decay and inductor stored energy

When an energised inductor is disconnected from its source but keeps a resistive current path, its stored magnetic energy drives the current while the resistor dissipates the energy.

Source-free current

The ODE is $L i' + Ri = 0$, and current continuity requires $i(0^+) = I_0$:

$$i(t) = I_0 e^{-t/\tau}, \quad \tau = \frac{L}{R}$$

Inductor voltage can reverse polarity

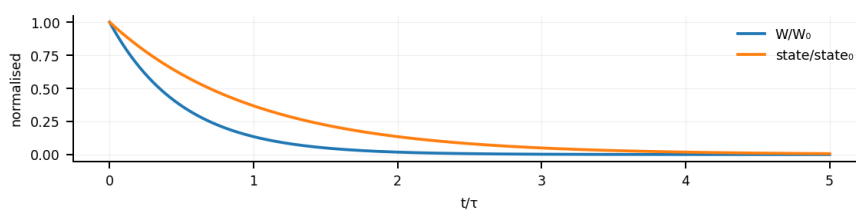
To keep the current flowing in the same direction just after switching, the inductor develops whatever voltage the loop requires. This is why abruptly opening an inductive path can create a large voltage.

$$v_L = L \frac{di}{dt} = -RI_0 e^{-t/\tau}$$

Energy dissipation

The stored energy decays as below; integrating Ri^2 over time returns exactly the initial stored energy when the resistor is the only loss path.

$$W_L(t) = \frac{1}{2} LI_0^2 e^{-2t/\tau}$$

**Worked example — stored energy and decay**

$L = 50\ \text{mH}$, $R = 10\ \Omega$ and $I_0 = 1.2\ \text{A}$, so $\tau = 5\ \text{ms}$ and $W_L(0) = 1/2 (0.05)(1.2^2) = 36\ \text{mJ}$. After 10 ms:

$$i = 1.2 e^{-2} = 0.162\ \text{A}, \quad W_L = 36 e^{-4} = 0.659\ \text{mJ}$$

7. Thévenin resistance and the time-constant shortcut

The time constant can often be found without re-deriving the ODE: look into the storage-element terminals with independent sources deactivated and find the equivalent resistance there.

Capacitor rule

Remove the capacitor, short independent voltage sources and open independent current sources, and find R_{th} at the capacitor terminals. Then $\tau = R_{th}C$.

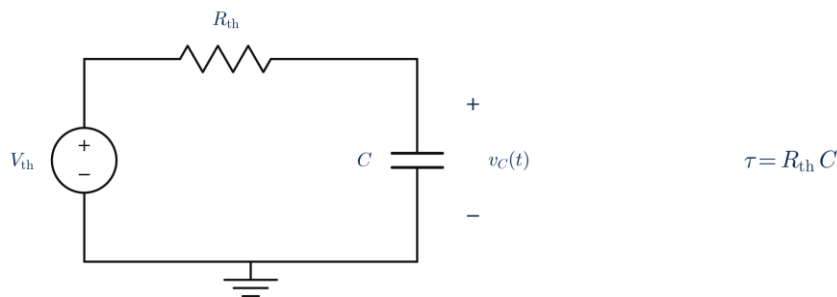
Inductor rule

Remove the inductor and find R_{th} at its terminals with the same rule. Then $\tau = L/R_{th}$.

Dependent sources need a test source

Dependent sources must stay active. Apply a test voltage or current at the port and calculate the ratio below; this detail is common in more advanced problems.

$$R_{th} = \frac{V_{test}}{I_{test}}$$



The capacitor sees a Thévenin source V_{th} behind R_{th} .

Worked example — a capacitor sees a resistor network

A capacitor sees $6\text{ k}\Omega$ to a 12 V source and $3\text{ k}\Omega$ to ground. With the source deactivated, the $6\text{ k}\Omega$ resistor also connects to ground:

$$R_{th} = 6\text{ k}\Omega \parallel 3\text{ k}\Omega = 2\text{ k}\Omega$$

For $C = 50\text{ }\mu\text{F}$, $\tau = 0.10\text{ s}$. The final voltage still comes from the active DC circuit: the shortcut gives τ , not $v(\infty)$.

8. RC low-pass step response: transient plus steady state

The RC charging circuit is also a first-order low-pass filter. Its step response separates naturally into a steady component and a transient that enforces the initial condition.

Transfer-function view

With the output across C , a step V_s/s gives the output transform below.

$$H(s) = \frac{1}{1 + sRC}, \quad V_o(s) = \frac{V_s}{s(1 + sRC)}$$

Complete response

The constant term is the zero-frequency steady response; the exponential is the transient belonging to the system pole.

$$v_o(t) = V_s(1 - e^{-t/RC})$$

Bandwidth–time connection

The pole angular frequency is $\omega_p = 1/(RC)$, so the cutoff frequency is given below. A faster time response needs a higher pole frequency, and therefore a wider bandwidth.

$$f_c = \frac{1}{2\pi RC}$$

Worked example — relate rise time to cutoff frequency

$R = 1\text{ k}\Omega$ and $C = 100\text{ nF}$ give $\tau = 100\text{ }\mu\text{s}$ and $f_c = 1/(2\pi\tau) = 1.59\text{ kHz}$. The 10–90% rise time of a first-order response is about $2.2\tau = 220\text{ }\mu\text{s}$, a useful bridge between transient specifications and filter design.

9. RL high-pass voltage response and initial/final behaviour

Taking the output across the inductor of a series RL circuit gives a high-pass response: a DC step produces an output that starts large and decays to zero.

Transfer function

With the output across L and $\tau = L/R$, there is a zero at the origin and a pole at $-1/\tau$:

$$H_L(s) = \frac{sL}{R + sL} = \frac{s\tau}{1 + s\tau}$$

Step response

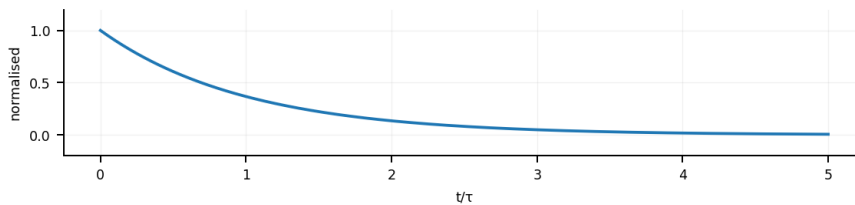
For $V_s u(t)$, $V_L(s) = V_s \tau / (1 + s\tau)$. The inductor initially opposes the change in current, so the full source voltage appears across it at $t = 0^+$.

$$v_L(t) = V_s e^{-t/\tau}$$

Complementary resistor output

The resistor voltage is the complement, so the two branch voltages sum to the source at every instant: a direct KVL check.

$$v_R(t) = V_s(1 - e^{-t/\tau})$$



Worked example — initial and final limits

$R = 20\text{ }\Omega$, $L = 0.20\text{ H}$ and $V_s = 10\text{ V}$, so $\tau = 10\text{ ms}$. $v_L(0^+) = 10\text{ V}$ and $v_L(\infty) = 0\text{ V}$. At $t = 10\text{ ms}$, $v_L = 10/e = 3.68\text{ V}$ and $v_R = 6.32\text{ V}$; their sum is still 10 V , confirming KVL.

10. Comparative summary and common exam errors

First-order RC and RL circuits share one mathematical template. Most exam mistakes come from state continuity, final values, the resistance seen by the storage element or sign conventions, not from difficult calculus.

$$x(t) = x_{\infty} + (x_0 - x_{\infty})e^{-t/\tau}$$

What is continuous?

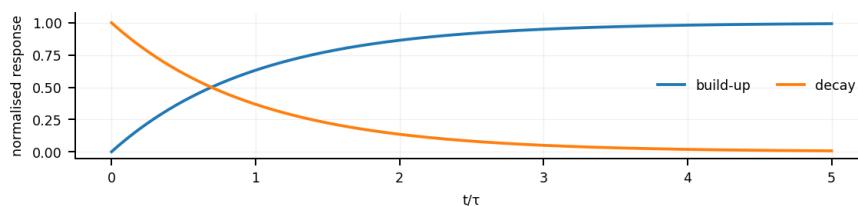
Capacitor voltage and inductor current are continuous under finite excitation. Resistor voltage and current may jump, and capacitor current and inductor voltage may jump to finite values when switching changes the circuit.

What sets the final value?

The post-switch DC equivalent: capacitor open, inductor short. Decide which post-switch circuit applies before setting derivatives to zero.

What sets the transient?

The difference $x(0^+) - x(\infty)$ sets the exponential coefficient; the time constant sets the decay rate. Find these two independently before writing the response. At $t = \tau$, 63.2% of the change is complete; by $t \approx 5\tau$, less than 1% of the initial exponential remains.



Exam checkpoint — diagnose four common errors

1. Allowing v_C or i_L to jump without an impulse.
2. Using the pre-switch network to find the post-switch final value.
3. Using the total resistance instead of the resistance seen by the storage element.
4. Writing $x_{\infty} + (x_0 + x_{\infty})e^{-t/\tau}$: the coefficient must be $x_0 - x_{\infty}$.