

Solving Circuit ODEs in the s-Domain

Ordinary Differential Equations for Electrical Engineers — Ch. 11 of 15

1. Standard Laplace transforms: exam reference sheet

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition / note
1	$\frac{1}{s}$	$\text{Re}(s) > 0$
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$n = 0, 1, 2, \dots$
e^{at}	$\frac{1}{s-a}$	$\text{Re}(s) > a$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	
$\cosh at$	$\frac{s}{s^2 - a^2}$	
$\sinh at$	$\frac{a}{s^2 - a^2}$	
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$	
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$	
$u(t-T)$	$\frac{e^{-sT}}{s}$	$T > 0$
$\delta(t-T)$	e^{-sT}	$T \geq 0$
$(t-T)u(t-T)$	$\frac{e^{-sT}}{s^2}$	
$(t-T)^n u(t-T)$	$\frac{n! e^{-sT}}{s^{n+1}}$	

Properties printed with most Laplace tables

Property	Transform / theorem
Linearity	$\mathcal{L}\{af + bg\} = aF + bG$
First derivative	$\mathcal{L}\{f'\} = sF - f(0^-)$
Second derivative	$\mathcal{L}\{f''\} = s^2F - sf(0^-) - f'(0^-)$
Integral	$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = F(s)/s$
Time delay	$\mathcal{L}\{f(t-T)u(t-T)\} = e^{-sT}F(s)$
Exponential shift	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Multiplication by t	$\mathcal{L}\{tf(t)\} = -\frac{dF}{ds}$
Convolution	$\mathcal{L}\{f * g\} = F(s)G(s)$
Initial value	$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$
Final value (check pole conditions first)	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$

The unilateral transform convention is assumed throughout.

2. Transforming a circuit ODE with non-zero initial conditions

The unilateral Laplace transform suits switching problems: differentiation becomes algebra, and the stored state just before switching appears explicitly in the transformed equation.

Transform the whole equation, not just the forcing

For $a y' + b y = x(t)$, the initial condition enters the algebra as a source term. For a second derivative, both initial values are carried into the s -domain.

$$a[sY(s) - y(0^-)] + bY(s) = X(s)$$

$$\mathcal{L}\{y''\} = s^2 Y - s y(0^-) - y'(0^-)$$

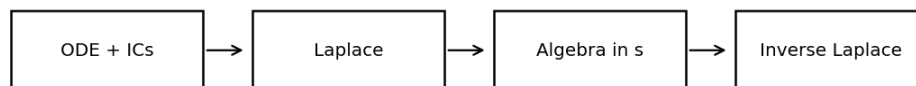
Solve for the transformed unknown

Collect every term multiplying $Y(s)$ before substituting a particular $X(s)$. This keeps the natural denominator visible and eases unit checks: the characteristic polynomial of the ODE reappears as the denominator.

$$Y(s) = \frac{X(s) + a y(0^-)}{as + b}$$

Interpret 0^- correctly

0^- denotes the state just before switching. For finite sources, capacitor voltage and inductor current are continuous, so these values usually equal the 0^+ values; an impulse at $t = 0$ is the important exception.



Worked example — first-order ODE with a stored initial state

Solve $0.2 y' + y = 6u(t)$ with $y(0^-) = 2$.

$$0.2[sY - 2] + Y = \frac{6}{s} \Rightarrow Y(0.2s + 1) = \frac{6}{s} + 0.4$$

$$Y(s) = \frac{30}{s(s+5)} + \frac{2}{s+5} = \frac{6}{s} - \frac{4}{s+5} \Rightarrow y(t) = 6 - 4e^{-5t}$$

The transform combines the forced response and the non-zero initial state automatically.

3. s-domain capacitor and inductor models with stored energy

Capacitors and inductors become algebraic elements in the s -domain, but their initial energy does not disappear: it appears as an extra source term whose sign follows from the chosen references.

Capacitor relation

From $i_C = C dv_C/dt$, $I_C = sC V_C - C v_C(0^-)$. Solving for voltage, the first term is the zero-state impedance drop and the second carries the initial capacitor voltage:

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

Inductor relation

From $v_L = L di_L/dt$, the zero-state impedance is sL and the term $-Li_L(0^-)$ represents the stored current:

$$V_L(s) = sL I_L(s) - L i_L(0^-)$$

Equivalent circuits depend on reference polarity

Textbooks draw the initial-condition sources in several equivalent orientations. Rather than memorising a diagram, derive the source sign from the transformed element law; this stays reliable when reference directions change. With zero initial state, $Z_C = 1/(sC)$ and $Z_L = sL$.

Capacitor

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

Inductor

$$V_L(s) = sL I_L(s) - L i_L(0^-)$$

Zero-state impedances

$$Z_C = \frac{1}{sC}$$

$$Z_L = sL$$

Worked example — initial capacitor voltage as an s-domain source

A 100 μF capacitor has $v_C(0^-) = 5 \text{ V}$ and carries $I_C(s) = 2/(s + 10) \text{ mA}$. With $1/(sC) = 10\,000/s \, \Omega$:

$$V_C(s) = \frac{I_C}{sC} + \frac{5}{s} = \frac{20}{s(s + 10)} + \frac{5}{s}$$

The $5/s$ term remains even with zero external current: it is the s-domain form of stored electric energy.

4. s-domain impedance and algebraic KCL/KVL

With zero initial conditions, the R, L and C laws are handled exactly like complex impedances, and Kirchhoff's laws become ordinary algebraic equations in s .

Impedance forms

These are not sinusoidal phasor impedances: s is a general complex variable, so the expressions describe transient as well as steady behaviour.

$$Z_R = R, \quad Z_L = sL, \quad Z_C = \frac{1}{sC}$$

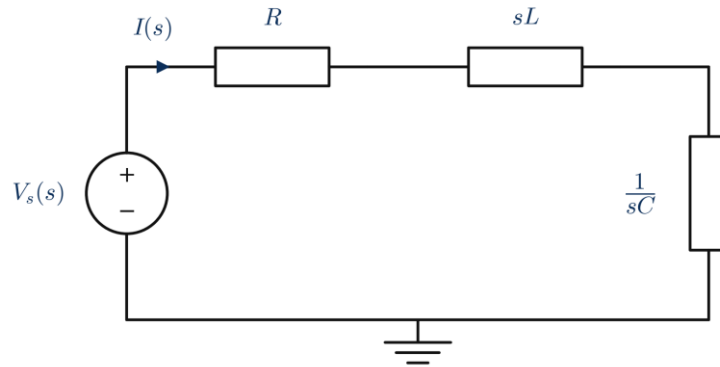
Series and parallel combinations

Series impedances add and parallel admittances add. Voltage and current division work, provided the zero-state assumption holds or initial-condition sources are included separately.

$$Z_{\text{series}} = R + sL + \frac{1}{sC}, \quad Y_{\text{parallel}} = \frac{1}{R} + \frac{1}{sL} + sC$$

KCL and KVL still express conservation

The transform changes only the element relations and the representation of sources, not the topology or Kirchhoff's laws. That is why complex switching networks reduce systematically to rational functions.



Series RLC circuit redrawn with s -domain impedances (zero initial state).

Worked example — series RLC current in the s -domain

For zero initial conditions and a step source V_s/s :

$$I(s) = \frac{V_s/s}{R + sL + \frac{1}{sC}} = \frac{V_s C}{LCs^2 + RCs + 1}$$

(multiplying top and bottom by sC). The same quadratic denominator appears in the time-domain RLC ODE.

5. Transfer functions $H(s)$ with zero initial conditions

A transfer function isolates the input–output dynamics of a linear time-invariant circuit. It is the ratio of output transform to input transform with all independent initial conditions set to zero.

Definition and meaning

$H(s)$ describes the zero-state response. It belongs to the circuit together with the chosen input and output, and it is not the complete response when energy is initially stored.

$$H(s) = \frac{Y(s)}{U(s)} \quad (\text{all initial conditions zero})$$

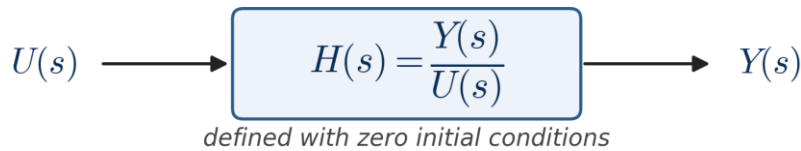
Numerator and denominator

After any algebraic cancellation, the denominator holds the system poles; the numerator sets the zeros and the relative weighting of the modes in the chosen output. Different outputs of one circuit can share the same poles but have different zeros.

Frequency response connection

For a stable system in sinusoidal steady state, evaluating $H(s)$ at $s = j\omega$ gives the frequency response. Phasor analysis is therefore a special case of Laplace analysis.

$$H(j\omega) = H(s)|_{s=j\omega}$$



Worked example — RC low-pass transfer function

For a series RC circuit with output across C : $V_o = I/(sC)$ and $V_i = I[R + 1/(sC)]$, so

$$H(s) = \frac{V_o}{V_i} = \frac{1}{1 + sRC}$$

With $R = 1 \text{ k}\Omega$ and $C = 10 \text{ }\mu\text{F}$, $H(s) = 1/(1 + 0.01s) = 100/(s + 100)$. The single pole at -100 s^{-1} corresponds to $\tau = 10 \text{ ms}$.

6. Poles, zeros, time constants, damping and stability

Pole-zero structure is the compact s -domain summary of the natural modes and the input-output shaping. For passive circuits with positive R, L and C , the natural poles lie in the open left half-plane.

$$H(s) = K \frac{\prod_k (s - z_k)}{\prod_k (s - p_k)}$$

Poles generate natural modes

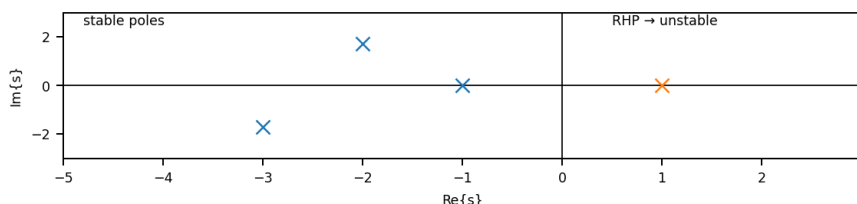
A simple real pole $p = -1/\tau$ contributes $e^{-t/\tau}$; a complex pair $-\alpha \pm j\omega_d$ contributes $e^{-\alpha t}[A\cos\omega_d t + B\sin\omega_d t]$. The real part controls growth or decay; the imaginary part controls oscillation.

Zeros shape what is observed

A zero can attenuate or cancel a mode in a particular transfer function. Exact pole-zero cancellation is an algebraic property; it does not remove the underlying physical state from the network.

Stability test

For a proper rational transfer function, asymptotic stability requires every uncanceled pole to have a negative real part. A right-half-plane pole gives exponential growth; imaginary-axis poles need separate interpretation, since persistent oscillation or integration may prevent a finite final value.



Worked example — infer the time-domain behaviour from the poles

A transfer function has poles at -50 and -200 s^{-1} and a zero at -100 s^{-1} . The natural response can contain e^{-50t} and e^{-200t} , and the -50 pole dominates late time. The time constants are 20 ms and 5 ms . Both poles are in the left half-plane, so the zero-state dynamics are asymptotically stable.

7. RC and RL transients by Laplace transform

First-order switching problems show the complete Laplace workflow in its simplest form: model, transform with initial conditions, solve a rational equation, decompose and invert.

RC step response

For $RC v_C' + v_C = V_s u(t)$ with $v_C(0^-) = V_0$, the transformed solution separates into a final-value term and a decaying pole at $-1/(RC)$:

$$V_C(s) = \frac{V_s/s + RC V_0}{RCs + 1}$$

RL current response

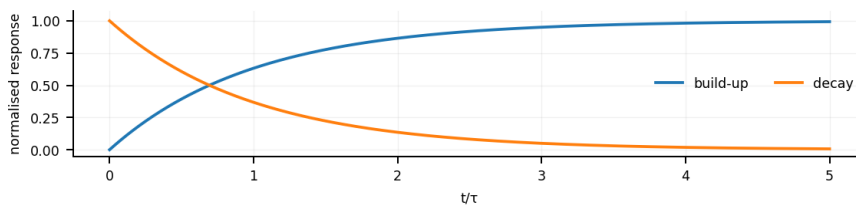
For $L i' + Ri = V_s u(t)$ with $i(0^-) = I_0$, the same algebra appears with $\tau = L/R$; the state is now a current:

$$I(s) = \frac{V_s/s + L I_0}{Ls + R}$$

The universal first-order form

Both results reduce to the form below. Laplace transformation derives it automatically, even when the source is delayed, piecewise or combined with other inputs.

$$x(t) = x_\infty + (x_0 - x_\infty)e^{-t/\tau}$$



Worked example — RC charging from zero

With $R = 2 \text{ k}\Omega$, $C = 50 \text{ }\mu\text{F}$, $V_s = 10 \text{ V}$ and $v_C(0^-) = 0$, $\tau = 0.10 \text{ s}$:

$$V_C(s) = \frac{10}{s(1 + 0.1s)} = \frac{10}{s} - \frac{10}{s + 10} \Rightarrow v_C(t) = 10(1 - e^{-10t}) \text{ V}$$

Checks: initial value $v_C(0^+) = 0$; final value $v_C(\infty) = 10 \text{ V}$.

8. RLC transients by Laplace transform

Second-order circuits are where Laplace methods really pay off: both initial conditions and the forcing enter one algebraic expression, and the denominator exposes damping and natural frequency before inversion.

Transform the capacitor-voltage ODE

For a series RLC with output v_C , $LC v_C'' + RC v_C' + v_C = v_s$. With $v_0 = v_C(0^-)$ and $v_0' = v_C'(0^-) = i(0^-)/C$:

$$LC[s^2V - sv_0 - v_0'] + RC[sV - v_0] + V = V_s(s)$$

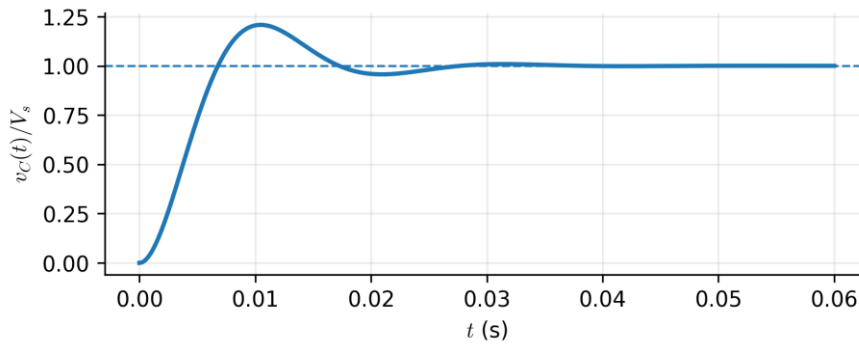
Factor or complete the square

After solving for $V_C(s)$, write the underdamped quadratic in the form $(s + \alpha)^2 + \omega_d^2$, which maps directly to decaying sine and cosine terms.

$$s^2 + 2\alpha s + \omega_0^2 = (s + \alpha)^2 + \omega_d^2$$

Separate the final and transient terms

A DC step contributes a pole at $s = 0$ from the source, alongside the circuit poles. The origin pole gives the constant final component v_{ss} ; the circuit poles give the transient v_n : $v_C(t) = v_{ss}(t) + v_n(t)$.



Worked example — underdamped 10 V step

Take $R = 20 \Omega$, $L = 0.10 \text{ H}$, $C = 100 \mu\text{F}$ and zero initial energy. Then $\alpha = 100 \text{ s}^{-1}$, $\omega_0 = 316.2 \text{ rad/s}$ and $\omega_d = 300 \text{ rad/s}$. For a 10 V step:

$$v_C(t) = 10 \left[1 - e^{-100t} \left(\cos 300t + \frac{1}{3} \sin 300t \right) \right] \text{ V}$$

The response starts at 0 V, rings about 10 V, and decays inside the envelope e^{-100t} .

9. Piecewise and delayed inputs using unit steps

The second shifting theorem represents switching events without solving a new differential equation on every interval: delays become factors e^{-sT} in the transform domain.

Represent a rectangular pulse

A pulse of amplitude A from T_1 to T_2 transforms as

$$A[u(t - T_1) - u(t - T_2)] \leftrightarrow \frac{A(e^{-sT_1} - e^{-sT_2})}{s}$$

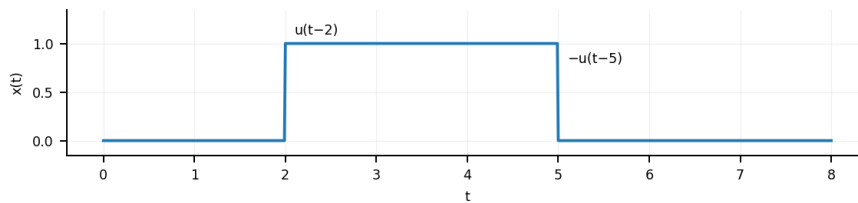
Shift the waveform itself

A delayed waveform is written $f(t - T)u(t - T)$, with transform $e^{-sT}F(s)$. The step is essential: the shifted waveform must be zero before its start time.

Response by superposition

In an LTI circuit, each delayed input term produces a correspondingly delayed copy of the zero-state response $g(t)$. Pulse and multi-switch problems become much cleaner than repeated interval-by-interval integration:

$$y_{\text{pulse}}(t) = A[g(t - T_1)u(t - T_1) - g(t - T_2)u(t - T_2)]$$



Worked example — RC response to a 3 s pulse

Let the normalised RC step response be $g(t) = 1 - e^{-t/\tau}$, and apply a unit pulse from $t = 2$ to $t = 5$:

$$y(t) = g(t - 2) u(t - 2) - g(t - 5) u(t - 5)$$

For $2 \leq t < 5$ only the charging term is active. After $t = 5$ the second shifted step subtracts the response that would have followed had the input stayed high. This one expression is valid for all t .

10. Convolution: the time-domain meaning of multiplication in s

Multiplying transforms corresponds to convolution in time. This connects transfer functions to impulse responses, and shows that an LTI output is a weighted accumulation of shifted impulse responses.

Convolution integral

If $Y(s) = H(s)X(s)$, then for causal signals the output is given below. Here λ is a dummy integration variable and t is the observation time.

$$y(t) = (h * x)(t) = \int_0^t h(t - \lambda) x(\lambda) d\lambda$$

Impulse response

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ is the output for the input $\delta(t)$ with zero initial conditions. Knowing $h(t)$ fixes the zero-state response to any admissible input through convolution.

Physical interpretation

At each earlier time λ , the input $x(\lambda)$ launches a scaled copy of $h(t - \lambda)$, and the present output is the sum of all these contributions. Laplace multiplication compresses this accumulation into algebra.

$$\begin{array}{c} x(\tau) \\ h(t - \tau) \end{array} \rightarrow \boxed{\text{overlap and integrate}} \rightarrow y(t) = \int_0^t x(\tau) h(t - \tau) d\tau$$

Worked example — RC step from its impulse response

For $H(s) = 1/(1 + s\tau)$, $h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$. With $x(t) = V u(t)$:

$$y(t) = \int_0^t \frac{1}{\tau} e^{-(t-\lambda)/\tau} V d\lambda = V(1 - e^{-t/\tau})$$

This is the familiar RC charging response.

11. Direct ODE or Laplace: when to use each

Direct time-domain and Laplace methods are not competing theories but two representations of the same linear dynamics. Choose the one that minimises algebra while preserving physical insight.

Use direct ODE methods when

The coefficients are simple, the forcing is elementary and the natural/particular split is easy to write. Direct methods often show physical time constants and modal structure with less algebra.

Use Laplace methods when

Initial conditions are non-zero, switching gives delayed or piecewise forcing, impulses are present, or transfer functions are needed. Laplace methods also make interconnecting subsystems straightforward: ODE $\rightarrow$ algebraic equation $\rightarrow$ inverse transform.

Always perform physical checks

Whichever route you use, verify dimensions, state continuity, initial slopes, final values, pole locations and energy behaviour. Both routes must give the same causal solution, and independent checks are worth more than symbolic complexity.

Direct time domain

best when ODE is simple
and forcing is elementary

Laplace domain

best for switching, ICs,
delays and rational algebra

same physical solution

Checkpoint — choose a method before calculating

A single RC driven by a DC step from zero is fastest with the standard first-order formula. The same RC driven by three delayed pulses is usually cleaner in Laplace form, since each delay becomes e^{-sT} . A second-order RLC with sinusoidal forcing can be solved either way; Laplace is attractive when non-zero initial energy must be included. Method choice should reduce work, not obscure the physics.