

The Laplace Transform: From ODEs to Algebra

Ordinary Differential Equations for Electrical Engineers — Ch. 9 of 15

1. Why Laplace transforms suit circuit transients

The unilateral Laplace transform turns time-domain differentiation into algebra in a complex variable s , while carrying initial conditions into the transformed equation. This makes it especially effective for switching transients and piecewise inputs.

Differential equations become algebraic equations

A derivative $f'(t)$ transforms to $sF(s) - f(0^-)$; higher derivatives give higher powers of s plus initial-condition terms. A linear ODE therefore becomes an algebraic equation for $Y(s)$, solved with ordinary algebra before transforming back.

$$\mathcal{L}\{f'\} = sF(s) - f(0^-)$$

Switching functions are handled cleanly

Steps, impulses and delayed functions have compact transforms. A delay introduces a factor e^{-sT} , replacing interval-by-interval integration with systematic algebra while preserving the switching time.

The transform does not remove the physics

The poles in the s -domain are the same natural modes as the characteristic roots in the time domain. Initial conditions still represent stored energy, and final-value checks still come from the physical circuit. Laplace analysis reorganises the calculation; it does not create a different model.

Deeper engineering point

The unilateral transform suits initial-value problems: it starts at the switching origin and brings in the pre-switch state through 0^- initial data. The bilateral transform used in wider signals-and-systems theory integrates over all time and needs explicit regions of convergence. For circuit transients the unilateral form is usually the more direct tool.



Worked example — transform a first-order IVP

For $y' + 3y = 6u(t)$ with $y(0^-) = 2$, transform term by term:

$$[sY - 2] + 3Y = \frac{6}{s} \Rightarrow (s + 3)Y = 2 + \frac{6}{s} \Rightarrow Y(s) = \frac{2}{s}$$

The cancellation shows that the initial value already equals the forced equilibrium $6/3 = 2$, so $y(t) = 2$ for $t \geq 0$, in agreement with the time-domain prediction.

2. Definition of the unilateral transform and the variable s

The unilateral transform integrates from the switching origin $t = 0^-$ to infinity, which suits causal circuit transients. The complex variable $s = \sigma + j\omega$ represents exponential growth or decay and sinusoidal rotation at the same time.

Definition

For a function of suitable growth:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

The kernel $e^{-st} = e^{-\sigma t} e^{-j\omega t}$ weights the waveform by decay and phase. Convergence requires σ to be large enough for the integral to stay finite.

Meaning of s

The real part σ sets the exponential weighting and the imaginary part ω the oscillation. A natural mode e^{pt} appears as a pole at $s = p$, because its transform is $1/(s - p)$ in its region of convergence.

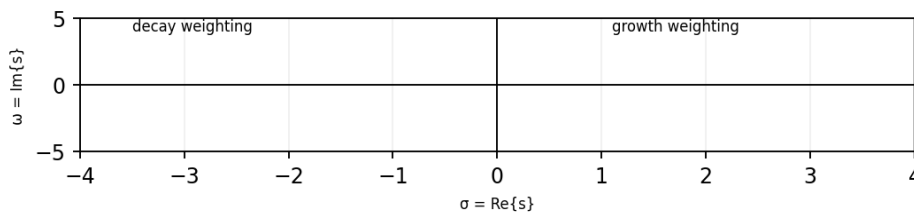
$$\mathcal{L}\{e^{at}\} = \frac{1}{s - a}, \quad \text{Re}\{s\} > \text{Re}\{a\}$$

Engineering units

Since st must be dimensionless, s has units s^{-1} . The transform itself carries one extra factor of time: the transform of a voltage has units V·s. This helps when checking transformed equations.

Deeper engineering point

In bilateral theory, the algebraic formula and its region of convergence together determine the signal. In unilateral circuit work, causality and the $t \geq 0$ convention usually fix the interpretation, so the rational expression gets most of the attention. Convergence still explains why $\text{Re}\{s\}$ must lie to the right of any fast-growing modes.



Worked example — evaluate the transform from the definition

Let $f(t) = e^{-2t}u(t)$. For $\text{Re}\{s\} > -2$:

$$F(s) = \int_0^{\infty} e^{-2t} e^{-st} dt = \int_0^{\infty} e^{-(s+2)t} dt = \left[\frac{-e^{-(s+2)t}}{s+2} \right]_0^{\infty} = \frac{1}{s+2}$$

The pole at $s = -2$ is the decay rate of the original waveform.

3. Transforms of constants, powers, exponentials and sinusoids

A small set of transform pairs covers most circuit inputs and response components. Rather than re-integrating every function, recognise the pair and check any shifts or scale factors carefully.

Basic polynomial pairs

The unit step transforms to $1/s$. More generally, for integer $n \geq 0$, a ramp $t u(t)$ gives $1/s^2$ and $t^2 u(t)$ gives $2/s^3$:

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Exponential pair

A decaying exponential e^{-at} with $a > 0$ gives $1/(s + a)$. This sign is a common source of error: a pole at $-a$ corresponds to the decay e^{-at} .

Sinusoidal pairs

The denominators have poles at $\pm j\omega$, the undamped sinusoidal modes. Multiplying by e^{-at} shifts these poles left to $-a \pm j\omega$.

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}, \quad \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Deeper engineering point

Transform tables are compressed results of the defining integral. Re-deriving a few central pairs (1 , e^{at} , sine and cosine) prevents sign errors and makes shifted forms easier to understand. Memorisation is far more reliable when each pair is linked to the pole pattern it produces in the s -plane.

$\mathcal{L}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
	1	e^{-at}	$\cos \omega t$	$\sin \omega t$
	$\frac{1}{s}$	$\frac{1}{s+a}$	$\frac{s}{s^2 + \omega^2}$	$\frac{\omega}{s^2 + \omega^2}$

Worked example — transform a mixed forcing function

Find $\mathcal{L}\{3 - 2e^{-4t} + 5\sin 6t\}$. By linearity:

$$F(s) = \frac{3}{s} - \frac{2}{s+4} + \frac{30}{s^2 + 36}$$

The poles reveal the component types: a DC term at $s = 0$, a decay pole at -4 , and sinusoidal poles at $\pm j6$.

4. Linearity, time scaling and time shifting

Transform properties let known pairs be reused for modified signals. Apply each property to the whole waveform, including its causal step factor, and keep amplitude and time scaling distinct.

Linearity

If $\mathcal{L}\{f\} = F$ and $\mathcal{L}\{g\} = G$, then the relation below holds. It is the transform counterpart of superposition, and it makes sums of step, exponential and sinusoidal inputs manageable.

$$\mathcal{L}\{af + bg\} = aF + bG$$

Time scaling

Compressing a waveform in time expands its s -domain scale and introduces a factor $1/a$. It is often safer to verify with a simple known pair than to rely on memory.

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0$$

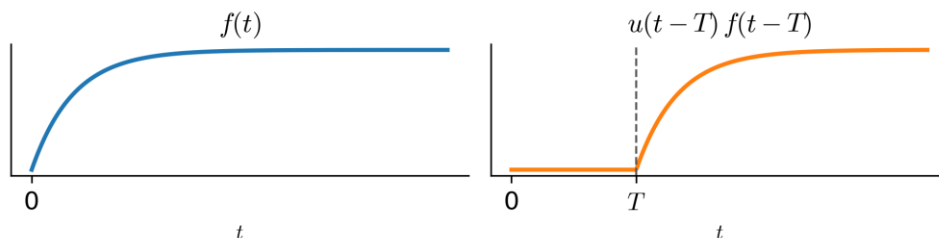
Delay and exponential shift

A delayed causal waveform picks up a factor e^{-sT} . Multiplying by e^{at} instead shifts the transform variable: $\mathcal{L}\{e^{at}f(t)\} = F(s - a)$. Delay adds an exponential factor; exponential weighting shifts the poles.

$$\mathcal{L}\{u(t - T) f(t - T)\} = e^{-sT} F(s)$$

Deeper engineering point

Do not confuse delay with exponential weighting. One changes when a waveform starts; the other changes its modal growth or decay rates. The algebra only looks similar because both involve exponentials.



Worked example — delay a known response

$f(t) = 1 - e^{-2t}$ has $F(s) = 1/s - 1/(s + 2)$. Delay it by 3 s and keep it zero before then:

$$g(t) = u(t - 3)[1 - e^{-2(t-3)}] \Rightarrow G(s) = e^{-3s} \left[\frac{1}{s} - \frac{1}{s + 2} \right]$$

Do not replace t by $t - 3$ without the step factor: the step enforces causality and is essential to the theorem.

5. Derivative and integral properties with initial conditions

The derivative property is the main reason Laplace transforms solve initial-value ODEs so efficiently. Integration by parts moves the derivative from the unknown onto the exponential kernel and exposes the initial value.

First derivative

The initial value appears explicitly, so a first-order ODE needs no separate constant-fitting step if the initial condition is inserted correctly during transformation.

$$\mathcal{L}\{f'\} = sF - f(0^-)$$

Higher derivatives

Applying the property again gives the second-derivative rule. Each derivative order adds a power of s plus the required initial data, mirroring the number of initial conditions the ODE needs.

$$\mathcal{L}\{f''\} = s^2F - sf(0^-) - f'(0^-)$$

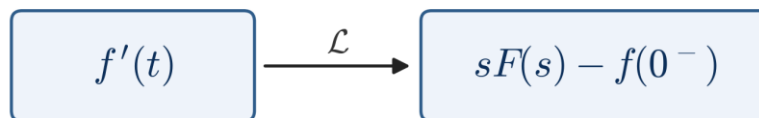
Time integration

If the integral starts from zero, division by s corresponds to accumulation. This is why step responses are related to impulse responses by a factor $1/s$.

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$$

Deeper engineering point

Using 0^- initial conditions is deliberate: the transformed equation describes what happens after the switch while carrying the stored state from just before it. If an impulse occurs exactly at $t = 0$, the distinction between 0^- and 0^+ is essential because the state may jump. Ordinary finite switching does not make capacitor voltage or inductor current jump.



Worked example — a non-zero initial condition enters automatically

Solve $y' + 2y = 4u(t)$ with $y(0^-) = 1$:

$$(sY - 1) + 2Y = \frac{4}{s} \Rightarrow Y = \frac{s + 4}{s(s + 2)} = \frac{2}{s} - \frac{1}{s + 2}$$

So $y(t) = 2 - e^{-2t}$. The initial condition entered through $sY - 1$; no constant-fitting was needed after inversion.

6. Unit-step, impulse, ramp and delayed functions

Generalised switching functions represent ideal turn-on, sudden injection of area and finite-duration forcing algebraically. They are especially useful when sources or topology change at known times.

Unit step and ramp

The unit step $u(t)$ is 0 for $t < 0$ and 1 for $t > 0$. Scaling and delaying these functions builds most practical piecewise waveforms.

$$\mathcal{L}\{u(t)\} = \frac{1}{s}, \quad \mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

Impulse

The Dirac impulse is defined by its sampling property $\int \delta(t) \phi(t) dt = \phi(0)$. In circuits it can represent an instantaneous transfer of charge or flux. It is a mathematical idealisation, not an infinitely tall ordinary pulse to be manipulated pointwise.

$$\mathcal{L}\{\delta(t)\} = 1, \quad \mathcal{L}\{\delta(t - T)\} = e^{-sT}$$

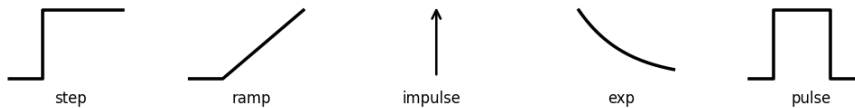
Delayed switching

$u(t - T)$ transforms to e^{-sT}/s . A pulse of height A from T_1 to T_2 is a difference of two delayed steps:

$$\mathcal{L}\{A[u(t - T_1) - u(t - T_2)]\} = \frac{A(e^{-sT_1} - e^{-sT_2})}{s}$$

Deeper engineering point

When its integral is dimensionless, the Dirac delta has units of inverse time, so $A\delta(t)$ has area A even though its pointwise height is not finite. An impulse current has units of A yet delivers a finite charge $\int i dt$. Thinking in terms of area avoids treating $\delta(t)$ like an ordinary huge number.



Worked example — transform a 4 V pulse

A 4 V pulse is present from $t = 1$ s to $t = 3$ s:

$$v(t) = 4[u(t - 1) - u(t - 3)] \quad \Rightarrow \quad V(s) = \frac{4(e^{-s} - e^{-3s})}{s}$$

The delay factors carry the two switching times. This form can be multiplied directly by a transfer function later, avoiding a separate ODE solution on each interval.

7. Initial- and final-value theorems and their validity

These theorems extract endpoint behaviour directly from $F(s)$, but they are conditional tools. The final-value theorem, in particular, must not be used blindly on unstable or sustained-oscillation responses.

Initial-value theorem

When the limits exist and there is no unaccounted impulse at $t = 0$, the theorem below applies. It is useful for checking that a transformed expression preserves a known capacitor voltage or inductor current.

$$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$$

Final-value theorem

If all poles of $sF(s)$ lie strictly in the open left half-plane (a simple pole of F at the origin may be cancelled by the factor s), then the final value exists and is given below. The pole condition is what guarantees the time-domain terms settle.

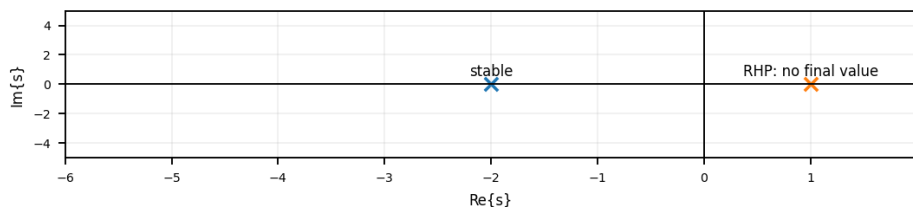
$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

When it fails

A right-half-plane pole causes growth; an imaginary-axis conjugate pair causes sustained oscillation. Neither has a finite final value, and obtaining a finite number from $\lim_{s \rightarrow 0} sF(s)$ does not override this. Always check the poles of $sF(s)$ first.

Deeper engineering point

The theorem works because stable exponential modes vanish as $t \rightarrow \infty$, leaving only the zero-frequency component. A right-half-plane pole grows instead, and an imaginary-axis pair keeps oscillating. The pole test is not a footnote: it is the statement that a final value exists at all.



Worked example — stable and unstable cases

For $F(s) = 5/[s(s+2)]$, the only pole of $sF = 5/(s+2)$ is in the left half-plane, so

$$f(\infty) = \lim_{s \rightarrow 0} \frac{5}{s+2} = 2.5$$

For $G(s) = 1/[s(s-1)]$, $sG = 1/(s-1)$ has a right-half-plane pole at $+1$. The theorem is invalid even though substituting $s = 0$ gives -1 : the actual response contains e^t and never settles.

8. Worked transforms of typical EE waveforms

Practical signals are usually assembled from the standard pairs and properties. The discipline is to rewrite the waveform in a recognisable causal form before transforming it.

Damped sinusoid

Apply the exponential shift to the sine pair. The poles $-a \pm j\omega$ show the decay rate and oscillation frequency directly.

$$\mathcal{L}\{e^{-at} \sin \omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

Delayed ramp

A ramp starting at T is $(t-T)u(t-T)$, not $t u(t-T)$: it starts from zero at $t = T$ instead of jumping to the value T .

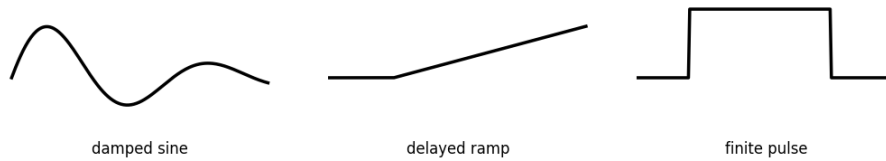
$$\mathcal{L}\{(t - T) u(t - T)\} = \frac{e^{-sT}}{s^2}$$

Finite pulse as two edges

A pulse is the difference of two delayed steps, so in a linear circuit its response is the difference of two delayed step responses.

Deeper engineering point

Decompose complicated waveforms into standard causal pieces before transforming. This mirrors superposition in the circuit and keeps every algebraic term interpretable: each has a clear origin, such as a delay, decay, sinusoid or switching edge.



Worked example — transform a switched damped sine

Let $f(t) = 3e^{-2(t-1)} \sin[4(t-1)]u(t-1)$. The undelayed waveform $g(t) = 3e^{-2t} \sin 4t$ has

$$G(s) = \frac{12}{(s+2)^2 + 16} \quad \Rightarrow \quad F(s) = \frac{12 e^{-s}}{(s+2)^2 + 16}$$

The factor e^{-s} is the 1 s delay; the denominator still encodes the damped oscillatory poles $-2 \pm j4$.

9. Laplace transform and property reference tables

Use these tables after understanding the derivations: recognise the time-domain pattern, match the pair, then check signs, shifts and initial-condition terms.

Time-domain $f(t)$	Laplace transform $F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$

Time-domain $f(t)$	Laplace transform $F(s)$
$u(t - T)$	$\frac{e^{-sT}}{s}$
$\delta(t - T)$	e^{-sT}
$(t - T)u(t - T)$	$\frac{e^{-sT}}{s^2}$

The main transform properties are summarised below.

Property	Result
Linearity	$\mathcal{L}\{af + bg\} = aF + bG$
Derivative	$\mathcal{L}\{f'\} = sF - f(0^-)$
Second derivative	$\mathcal{L}\{f''\} = s^2F - sf(0^-) - f'(0^-)$
Integral	$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = F/s$
Time delay	$\mathcal{L}\{u(t - T)f(t - T)\} = e^{-sT}F(s)$
Exponential shift	$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$
Initial value	$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$
Final value (poles of sF in left half-plane)	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$

Quick use — transform a second derivative correctly

If $y(0^-) = 2$ and $y'(0^-) = -1$, then

$$\mathcal{L}\{y''\} = s^2Y - sy(0^-) - y'(0^-) = s^2Y - 2s + 1$$

The final "+1" is easy to miss: the formula subtracts $y'(0^-)$, and that value is itself -1 . Sign discipline is essential.

10. Consolidation and degree-level checkpoint

Laplace competence means more than recalling a table. You should be able to choose a property, carry initial conditions, interpret poles, and reject results that violate time-domain physics.

A repeatable transform workflow

Write the causal waveform clearly. Apply linearity and standard pairs, then delay or exponential-shift properties as needed. For ODEs, transform every derivative with its initial-condition terms before collecting $Y(s)$.

Algebra before inversion

Solve for $Y(s)$, factor the denominator and simplify. The poles should match the expected natural modes of the ODE or circuit; delays should appear as e^{-sT} factors, not as unexplained changes in the denominator.

Endpoint and unit checks

Use the initial- and final-value theorems only when their conditions hold. Check units and compare the final value with the DC circuit where appropriate. A transform solution should strengthen physical interpretation, not replace it.

Deeper engineering point

After solving for $Y(s)$, compare its poles with the characteristic roots of the original ODE. They should agree unless a pole is cancelled by a numerator factor in that particular output. This links the transform algebra back to time-domain dynamics and catches accidental cancellations and sign errors.

**Worked checkpoint — first-order ODE from start to finish**

Solve $y' + 4y = 8u(t)$ with $y(0^-) = 0$. Transforming, $sY + 4Y = 8/s$, so

$$Y = \frac{8}{s(s+4)} = \frac{2}{s} - \frac{2}{s+4} \Rightarrow y(t) = 2(1 - e^{-4t})u(t)$$

Checks: $y(0^+) = 0$; the final-value theorem gives 2; the pole at -4 is stable; and substituting $y' = 8e^{-4t}$ gives $y' + 4y = 8$ for $t > 0$.