

General ODE Modelling of Lumped Electrical Networks

Ordinary Differential Equations for Electrical Engineers — Ch. 8 of 15

1. Choosing variables and writing independent KCL/KVL

Beyond a single loop or node, the main modelling difficulty is not calculus but choosing variables and a non-redundant set of equations. A good choice of variables makes the dynamic structure visible.

Node-voltage and mesh-current viewpoints

Node analysis is usually efficient when many elements meet at a few nodes; mesh analysis suits planar circuits with few loops. Either works if polarities and current directions are consistent. The goal is a set of independent equations, not the largest possible set.

$$\text{KCL: } A \mathbf{i}_b = \mathbf{0} \quad \text{KVL: } B \mathbf{v}_b = \mathbf{0}$$

Conceptually, A collects the independent node balances and B the independent loop balances of the branch currents $\mathbf{i}_b$ and branch voltages $\mathbf{v}_b$.

Prefer variables tied to stored energy

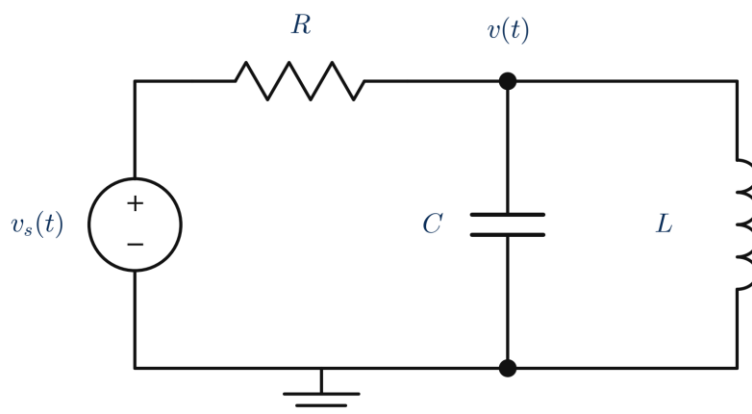
Capacitor voltages and inductor currents are the natural dynamic variables: they are continuous states, and their derivatives appear directly in the element laws. Resistor currents and voltages can usually be eliminated algebraically once KCL/KVL is written.

Avoid redundant conservation equations

In a connected network the node equations are not all independent, since they sum to zero; loop equations can also be dependent. Redundancy adds no physics, and it can create singular algebra and hide the true system order.

Deeper engineering point

The independence of KCL and KVL equations has a graph-theoretic basis. A connected circuit with N nodes has $N - 1$ independent node equations; the last one adds nothing. Similar dependencies exist among loop equations. Circuit simulators exploit this automatically, and hand analysis benefits from the same awareness.



Source and resistor feeding a node with C and L to ground.

Worked example — choose a minimal variable set

A source drives R in series with a node that has C and L to ground. Let $v(t)$ be the node (capacitor) voltage and $i_L(t)$ the inductor current.

$$\frac{v_s - v}{R} = C v' + i_L, \quad v = L i_L'$$

These are two independent first-order equations in two states. A separate resistor-current unknown is unnecessary, since it is simply $(v_s - v)/R$.

2. Reducing coupled equations to a single higher-order ODE

A multi-state network can be represented by coupled first-order equations or by one higher-order scalar ODE. Elimination changes the mathematical form, not the underlying modes.

Start from a coupled physical model

For a series RLC circuit, a convenient pair makes the physical states explicit and is often preferable for simulation:

$$v_C' = \frac{i}{C}, \quad i' = \frac{v_s - Ri - v_C}{L}$$

Eliminate one variable for a scalar ODE

Differentiate $v_C' = i/C$ to get $v_C'' = i'/C$, substitute i' from KVL and replace i by Cv_C' :

$$LC v_C'' + RC v_C' + v_C = v_s$$

Elimination can introduce source derivatives

If current is chosen instead, differentiating may produce v_s' . This is valid for smooth sources but needs care for steps, whose derivative contains an impulse. Where possible, choose the scalar variable that avoids differentiating the source.

Deeper engineering point

A scalar ODE is convenient for analysis but may have large coefficient ranges or input derivatives that are undesirable numerically. Coupled first-order equations preserve physical scaling and are usually preferred for simulation. Algebraic compactness and numerical quality are not the same objective.



Worked example — eliminate i_L from a parallel RLC model

KCL gives $i_s = Cv' + v/R + i_L$, and the inductor gives $i_L' = v/L$. Differentiating KCL, $i_s' = Cv'' + v'/R + i_L'$, and substituting:

$$C v'' + \frac{1}{R} v' + \frac{1}{L} v = i_s'$$

The equation is second order because the original model had two independent states, v and i_L .

3. State variables: capacitor voltages and inductor currents

State variables are the smallest set of quantities whose present values, together with the future input, determine the future evolution of the ideal lumped model. In RLC networks they are the stored-energy variables.

What makes a state variable special

A state is not just any signal. A resistor voltage can be computed instantly from the state and input, so it needs no initial condition of its own; capacitor voltage and inductor current carry memory of the past.

State vector notation

Collect the n independent states in a vector. For a circuit with two capacitors and one inductor, one possible state vector is $[v_{c1}, v_{c2}, i_L]^T$, provided no topology constraint makes a state dependent on the others.

$$\mathbf{x}(t) = [v_{c1} \ \cdots \ v_{cm} \ i_{L1} \ \cdots \ i_{Ln}]^T$$

Order and independent storage elements

Once algebraic constraints are removed, the number of independent storage states usually equals the system order. This is a useful check: a model claiming third-order dynamics from one capacitor and one inductor deserves scrutiny.

Deeper engineering point

A state vector is not unique. Any invertible linear transformation $\mathbf{z} = T\mathbf{x}$ gives an equivalent description with the same input–output behaviour and poles. Capacitor voltages and inductor currents are preferred because they are physically meaningful, but control theory often chooses transformed states that simplify the matrices or reveal modes.

$$\mathbf{x}(t) = [v_{C1} \ v_{C2} \ \cdots \ i_{L1} \ i_{L2}]^T$$

Worked example — identify states in a two-storage network

For a series RLC circuit choose $x_1 = v_C$ and $x_2 = i$:

$$x_1' = \frac{x_2}{C}, \quad x_2' = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}v_s$$

No resistor state is needed: once x_1 and x_2 are known, $v_R = Rx_2$ and $v_L = v_s - v_R - x_1$ are algebraic outputs. The two equations need the two physical initial values $v_C(0)$ and $i(0)$.

4. State-space form $\mathbf{x}' = A\mathbf{x} + B\mathbf{u}$

State space keeps an n th-order system as n coupled first-order equations. It is the natural language of simulation, control and numerical integration, while staying directly connected to KCL, KVL and the element laws.

Matrix form

For a linear time-invariant circuit, A contains the internal state coupling and therefore sets the natural poles; B describes how inputs drive the states; C and D select and combine outputs.

$$\mathbf{x}' = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

Series-RLC example

With $\mathbf{x} = [v_C, i]^T$ and input $u = v_s$, the coefficients of the state equations form the rows of A and B directly:

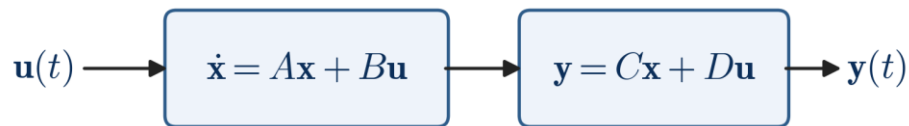
$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

Why this representation is useful

State space avoids differentiating discontinuous inputs, handles multiple inputs and outputs naturally, and maps directly to time-stepping solvers. The price is a vector-valued solution instead of one familiar scalar equation. The eigenvalues of A are the natural poles.

Deeper engineering point

For an LTI system the homogeneous state solution is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$. The eigenvalues of A are the natural poles and the eigenvectors are the modal state directions. Matrix exponentials are not needed for every introductory circuit, but this explains why state-space and characteristic-equation methods give the same modal rates.



Worked example — build the numerical A matrix

Let $R = 10\ \Omega$, $L = 20\ \text{mH}$ and $C = 100\ \mu\text{F}$, so $1/C = 10^4$, $1/L = 50$ and $R/L = 500\ \text{s}^{-1}$. With $\mathbf{x} = [v_C, i]^T$:

$$A = \begin{bmatrix} 0 & 10\,000 \\ -50 & -500 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

The characteristic equation is $\det(sI - A) = s^2 + 500s + 5 \times 10^5$, exactly matching the scalar capacitor-voltage ODE of the same circuit.

5. Response terminology: zero-input, natural, transient and more

Several response decompositions are used in circuits and systems. They overlap but are not identical, so precise terminology prevents conceptual errors.

Zero-input and zero-state

The zero-input response comes from initial stored energy with the input set to zero; the zero-state response comes from the input with all initial states zero. For a linear system, complete response = zero-input + zero-state.

Natural and forced

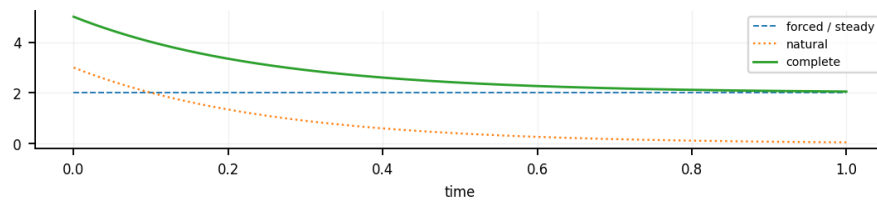
The natural response is built from the homogeneous modes (the poles); the forced response is a particular response associated with the input. With non-zero initial conditions, the natural part carries the constants needed to match the initial state. Complete response = natural + particular.

Transient and steady state

"Transient" describes components that vanish at large times in a stable system; "steady state" is what remains. A forced term can itself be transient (for example, a decaying exponential input), so "forced" does not always mean "steady state". Likewise, an unstable natural mode is not transient, because it does not decay.

Deeper engineering point

The terminology depends on the decomposition. Zero-input/zero-state separates initial conditions from inputs; natural/forced separates homogeneous from particular components; transient/steady state describes long-time behaviour. One term can belong to several categories, so use the pair that answers the question being asked.



Worked example — label the pieces correctly

For $y' + 2y = 4$ with $y(0) = 5$, the complete solution is $y = 2 + 3e^{-2t}$. The constant 2 is the particular (forced) steady-state response; $3e^{-2t}$ is natural and transient.

If the input were $4e^{-t}$ instead, the particular response would also decay: still forced, but not a non-zero steady state. This distinction matters for more advanced input waveforms.

6. Common input models: impulse, step, ramp and more

Engineering inputs are usually represented by a small library of idealised waveforms. They make switching and forcing explicit, and become especially convenient in Laplace analysis.

Step, ramp and impulse

The unit step $u(t)$ models abrupt turn-on; the ramp $r(t) = t u(t)$ grows linearly after switching. The Dirac impulse $\delta(t)$ is not an ordinary function: it is an idealised input of unit area, and the derivative of $u(t)$ in the distributional sense.

$$u(t), \quad \delta(t) = \frac{du}{dt}, \quad r(t) = t u(t)$$

Exponentials and sinusoids

$Ae^{-at}u(t)$ models a decaying source or envelope; $A\cos(\omega t + \phi)u(t)$ models a sinusoid switched on at $t = 0$. LTI circuits respond to these in highly structured ways, because exponentials and complex exponentials are closely related to natural modes.

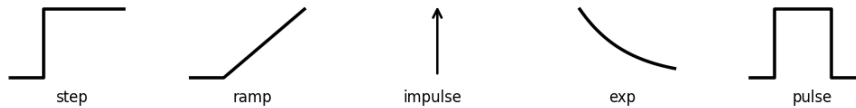
Finite pulses

A rectangular pulse of amplitude A from 0 to T is a difference of shifted steps, so by linearity its response is the difference of two step responses.

$$A[u(t) - u(t - T)]$$

Deeper engineering point

Ideal steps and impulses are mathematical limits: real sources have finite rise time and bandwidth. They are valuable because the response of a linear system to realistic signals can be built from them. The impulse response is especially important: through convolution it completely characterises a causal LTI system.



Worked example — express a delayed rectangular pulse

A 5 V pulse starts at $t = 2$ ms and ends at $t = 7$ ms:

$$v_s(t) = 5 u(t - 2 \text{ ms}) - 5 u(t - 7 \text{ ms})$$

The first term turns the source on; the second subtracts the same amplitude at turn-off. In a linear circuit the total response is the superposition of the responses to the two delayed steps.

7. Piecewise switching and resetting initial conditions

When the topology or sources change more than once, solve interval by interval. The final state of one interval is the initial state of the next, subject to capacitor-voltage and inductor-current continuity.

Define each interval explicitly

Write the circuit and ODE for $0 \leq t < t_1$, then separately for $t \geq t_1$. Connections can change, so time constants and final values can change too; one formula with one τ is not generally valid across the switch.

$$x(t_1^+) = x(t_1^-) \quad \text{for capacitor voltage and inductor current}$$

Carry the state, not the old equation

Evaluate the state at the end of the first interval and use it as $v_C(t_1^+)$ or $i_L(t_1^+)$ in the second. Using local time $\theta = t - t_1$ keeps exponentials readable and avoids delay sign errors.

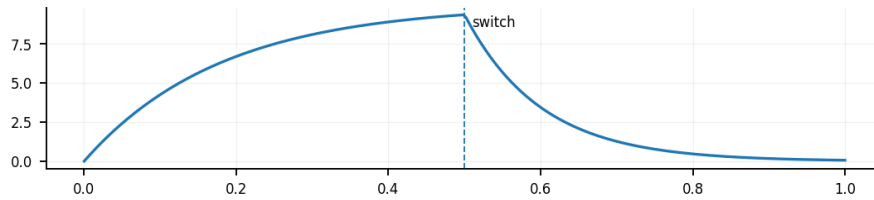
$$x_2(t) = x_{f2} + [x(t_1) - x_{f2}]e^{-(t-t_1)/\tau_2}$$

Check continuity and the new slope

For finite excitation the state is continuous, but its derivative can jump because the circuit seen by the storage element changes instantly. Apply the new KCL/KVL at t_1^+ to find the new derivative and confirm the second-interval response starts in the right direction.

Deeper engineering point

A switched circuit is a simple hybrid dynamical system: continuous state evolution interrupted by discrete changes of topology or forcing. The state usually stays continuous while the differential equation changes. This view scales naturally to power converters, relay systems and sampled switching networks.



Worked example — charge, then discharge

An RC with $\tau_1 = 0.10$ s charges from 0 towards 10 V for $0 \leq t < 0.20$ s:

$$v_C(0.20) = 10(1 - e^{-2}) = 8.647 \text{ V}$$

At $t = 0.20$ s the source is removed and the capacitor discharges through a different resistance with $\tau_2 = 0.05$ s. For $t \geq 0.20$ s:

$$v_C(t) = 8.647 e^{-(t-0.20)/0.05} \text{ V}$$

The voltage is continuous at 0.20 s even though its derivative changes abruptly.

8. Dependent sources and op-amp circuits that produce ODEs

Controlled sources and ideal op-amps do not remove dynamics; they change the algebraic constraints around the capacitors and inductors. The same KCL/KVL-plus-element-law workflow applies.

Dependent sources remain active

A source whose value depends on another circuit voltage or current is part of the circuit law. When finding a time constant from the resistance seen by a storage element, dependent sources cannot be "turned off"; use a test source if an equivalent resistance is needed.

Ideal op-amp assumptions

For an ideal op-amp with negative feedback, the input currents are zero and the input terminal voltages are equal. Combined with KCL at the summing node and the capacitor law, these constraints produce the ODEs of active filters and integrators.

$$i_+ = i_- = 0, \quad v_+ = v_- \quad (\text{linear, negative feedback})$$

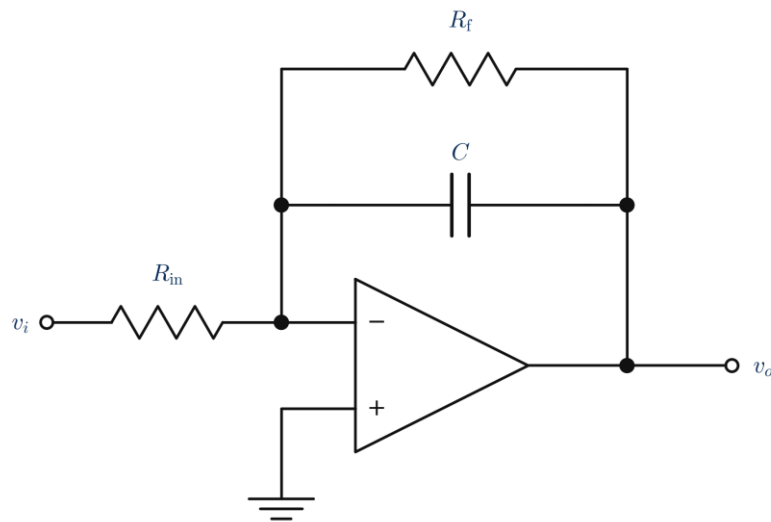
Example active first-order network

For an inverting stage with input resistor R_{in} and feedback R_f in parallel with C , the inverting node is a virtual ground, and KCL gives a stable first-order low-pass equation:

$$C \frac{dv_o}{dt} + \frac{v_o}{R_f} = -\frac{v_i}{R_{in}}, \quad \tau = R_f C, \quad \text{DC gain} = -\frac{R_f}{R_{in}}$$

Deeper engineering point

The ideal op-amp assumes unlimited gain, bandwidth and output capability. Real devices add finite gain-bandwidth, slew-rate limits, input bias currents and saturation, which can introduce extra states and poles. The ideal first-order ODE is a low-frequency model whose validity must be checked against the device and signal levels.



Inverting active low-pass stage: R_f in parallel with C in the feedback path.

Worked example — active low-pass step response

Let $R_{in} = 10 \text{ k}\Omega$, $R_f = 50 \text{ k}\Omega$ and $C = 2 \text{ }\mu\text{F}$. Then $\tau = R_f C = 0.10 \text{ s}$ and the DC gain is -5 . For a 1 V step with $v_o(0) = 0$, the final output is -5 V :

$$v_o(t) = -5[1 - e^{-t/0.10}] \text{ V}$$

The output starts moving negative. From the ODE, $v_o'(0) = -(1 \text{ V})/(R_{in}C) = -50 \text{ V/s}$, matching the derivative of the solution.

9. Checks: causality, continuity, slope, final value, energy

A circuit model should be challenged from several independent directions. Verification catches mistakes that symbolic manipulation alone can miss.

Causality and switching

A causal circuit initially at rest cannot respond before its input is applied: an input at t_0 cannot affect the output for $t < t_0$. A formula containing $u(t + t_0)$ for a source that turns on at $+t_0$ is suspicious; delayed inputs give delayed, not advanced, responses.

Continuity and initial slope

Finite sources preserve capacitor voltage and inductor current across switching. With those states fixed, KCL/KVL gives the initial derivative. A candidate with the right initial value but the wrong slope usually has a sign or time-constant error.

Final value and energy

Use the DC equivalent circuit for constant long-time inputs. In passive source-free circuits, stored energy cannot increase: $dW/dt \leq 0$. Also check dimensions and, where practical, substitute the solution into the original ODE. Several checks are stronger because they fail for different kinds of mistake.

Deeper engineering point

Verification is strongest when the checks are independent. Substituting into the same algebra used to derive a solution can repeat a hidden sign error, whereas an energy or DC-final-value check uses different physical reasoning. In demanding work, combine symbolic checks, limiting cases and numerical simulation.

**Worked example — reject an attractive but impossible answer**

An RC capacitor starts at 4 V and discharges source-free with $\tau = 0.20$ s. A proposed result is $v_C = 4e^{+5t}$. It satisfies $v_C(0) = 4$ V, but it grows, so the resistor would have to create energy, contradicting passive behaviour.

Substitution also exposes it: the natural ODE is $v' + 5v = 0$, but $v' = +5v$ gives $v' + 5v = 10v \neq 0$. One correct check would have been enough; several make the diagnosis immediate.

10. Consolidation: from schematic to ODE

General network modelling is a disciplined sequence: define variables, write independent conservation laws, insert element relations, eliminate only what is necessary, then verify the dynamic order and the physics.

From schematic to equations

Mark node voltages, branch currents, source polarities and stored-energy states. Choose independent node or mesh equations, and keep them in physical form long enough to see which term came from which element.

From equations to dynamic form

Substitute the R , C and L laws. Either keep a coupled state-space model or eliminate variables to get a scalar ODE. Normalise the coefficients, check units, and compare the order with the number of independent storage states.

From model to trusted result

Find pre-switch states, continuity conditions, initial derivatives and long-time limits. Solve with a method suited to the equation and input. Finally check causality, the original equations, units and energy. A neat closed form is useless if it describes the wrong circuit.

Deeper engineering point

State-space and scalar ODE models are complementary. State space is usually better for many states, multiple inputs and numerical integration; a scalar ODE is often better for hand analysis and classical solution methods. A skilled modeller moves between them and chooses whichever makes the current question easiest.



Worked checkpoint — state model and scalar model agree

For a series RLC choose $x_1 = v_C$ and $x_2 = i$. The state model is $x_1' = x_2/C$ and $x_2' = -(1/L)x_1 - (R/L)x_2 + v_s/L$. Differentiating x_1' and substituting x_2' :

$$x_1'' = -\frac{1}{LC}x_1 - \frac{R}{L}x_1' + \frac{v_s}{LC}$$

Multiplying by LC gives $LC v_C'' + RC v_C' + v_C = v_s$. The matrix and scalar descriptions are two representations of the same physical model.