

Second-Order Circuit ODEs: RLC Networks

Ordinary Differential Equations for Electrical Engineers — Ch. 7 of 15

1. Why two storage elements create second-order dynamics

An RLC circuit can store energy in both an electric and a magnetic field. When the capacitor and inductor states are independent, two initial conditions are needed and the natural dynamics are generally second order.

Two states, two initial conditions

The natural state variables are the capacitor voltage v_C and the inductor current i_L ; a resistor has no independent state. If the topology lets both vary independently, the system has two dynamic degrees of freedom, and eliminating one state leaves a scalar equation with a second derivative.

$$W(t) = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2$$

Energy exchanges before it is dissipated

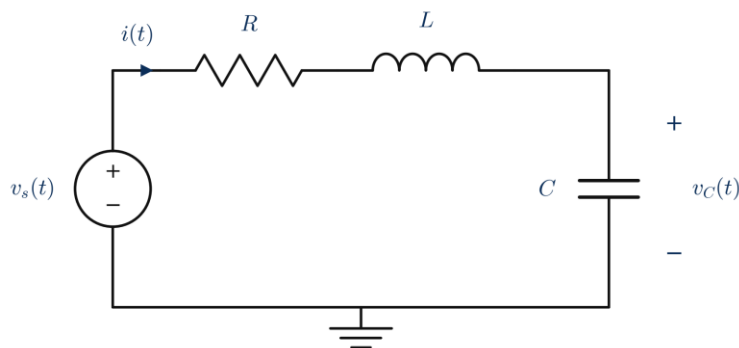
In a lossless LC circuit the total energy shuttles between the capacitor and the inductor, producing sustained oscillation. Adding R dissipates energy, so the oscillation decays or disappears depending on the level of damping.

Topology can reduce the apparent order

Counting components is not always enough. Two capacitors directly in parallel share one voltage and give only one independent state; two inductors in series can share one current. System order follows the independent stored-energy states, not the raw component count.

Deeper engineering point

The link between storage elements and order is structural. Network constraints can make nominal storage variables dependent; conversely, coupled subsystems can create higher-order behaviour even when the measured output hides some modes. State-space modelling later gives a precise way to count independent states and to spot a scalar ODE that has lost or cancelled a mode.



Series RLC circuit: one capacitor voltage and one loop current.

Worked example — count states before deriving equations

A series RLC circuit has one capacitor and one inductor, with the same loop current through all elements. The capacitor voltage and loop current can start independently, so two state quantities are needed. With v_C as the unknown, $i = C v_C'$ and

$$v_L = L \frac{di}{dt} = LC v_C''$$

KVL therefore contains v_C'' , confirming second-order dynamics. The two initial conditions can be supplied as $v_C(0)$ and $i(0)$, with $v_C'(0) = i(0)/C$.

2. The series-RLC ODE for current and capacitor voltage

A series RLC circuit can be described by either the loop current or the capacitor voltage. The best choice depends on the forcing and the output of interest, but both descriptions have the same poles.

Capacitor-voltage equation

KVL gives $v_s = v_L + v_R + v_C$. With $i = C v_C'$ and $v_L = L di/dt = LC v_C''$:

$$LC v_C'' + RC v_C' + v_C = v_s(t)$$

Dividing by LC reveals the standard second-order coefficients.

Current equation

Starting from $v_s = Li' + Ri + \frac{1}{C} \int i dt$ and differentiating gives

$$L i'' + R i' + \frac{1}{C} i = \frac{dv_s}{dt}$$

This suits problems where current is the output, but differentiating the source means impulses at switching edges must be handled carefully.

Same characteristic equation

For the source-free circuit, both formulations lead to the same characteristic equation, so the natural modes do not depend on the chosen variable; only the modal coefficients differ.

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

Deeper engineering point

Differentiating a circuit equation can simplify elimination but changes how discontinuous sources appear. A voltage step has an impulsive derivative, so the current ODE containing v_s' must be read in a distributional sense at the switching instant. The capacitor-voltage ODE avoids this and is usually cleaner for step-driven series RLC transients.

Worked example — derive and normalise the capacitor-voltage model

Let $L = 20 \text{ mH}$, $R = 10 \Omega$ and $C = 100 \mu\text{F}$. Then $LC = 2 \times 10^{-6} \text{ s}^2$ and $RC = 1 \times 10^{-3} \text{ s}$:

$$(2 \times 10^{-6})v_C'' + (1 \times 10^{-3})v_C' + v_C = v_s$$

Dividing by LC :

$$v_C'' + 500 v_C' + 5.0 \times 10^5 v_C = 5.0 \times 10^5 v_s$$

The coefficients have units s^{-1} and s^{-2} as required, and damping and natural frequency are now visible.

3. The parallel-RLC ODE using node voltage

In a parallel RLC network the common node voltage is the natural scalar variable. KCL gives an equation containing the inductor current; differentiating once eliminates it and gives a second-order voltage ODE.

Start with KCL

For a current source i_s feeding parallel R , L and C branches, with $v = L i_L'$:

$$i_s = C v' + \frac{v}{R} + i_L$$

Differentiate to eliminate i_L

Differentiating gives $i_s' = C v'' + v'/R + i_L'$, and substituting $i_L' = v/L$:

$$C v'' + \frac{1}{R} v' + \frac{1}{L} v = \frac{di_s}{dt}$$

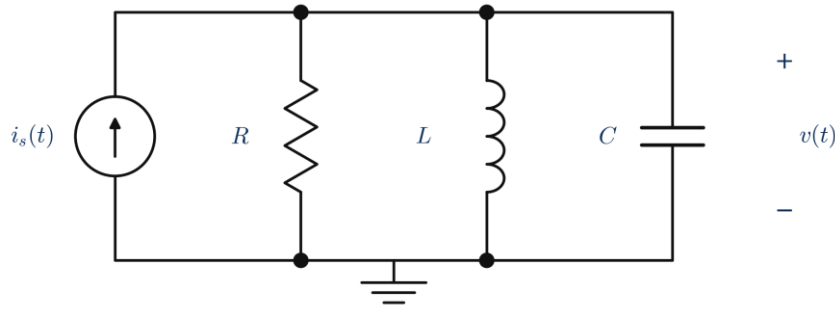
$$v'' + \frac{1}{RC} v' + \frac{1}{LC} v = \frac{i_s'}{C}$$

Series and parallel damping differ

The term $1/(LC)$ is the same ideal LC exchange rate, but the damping coefficient depends on topology. In a series RLC, increasing R increases damping; in a parallel RLC, increasing R reduces the conductance $1/R$ and therefore reduces damping.

Deeper engineering point

The same warning applies to a current-step-driven parallel RLC: differentiating KCL introduces i_s' , which contains an impulse at the switching edge. A two-state formulation in v and i_L represents the switch without differentiating the source. The choice of representation can improve both clarity and numerical robustness.



Parallel RLC circuit driven by a current source.

Worked example — identify the natural coefficients

For $R = 200 \, \Omega$, $L = 50 \, \text{mH}$ and $C = 20 \, \mu\text{F}$, $1/(RC) = 250 \, \text{s}^{-1}$ and $1/(LC) = 1.0 \times 10^6 \, \text{s}^{-2}$, so

$$v'' + 250 v' + 10^6 v = 0$$

The natural-frequency scale is $1000 \, \text{rad/s}$, and the damping factor is half the v' coefficient: $\alpha = 125 \, \text{s}^{-1}$.

4. Natural frequency ω_0 and damping factor α

The standard second-order form separates two physical effects: the LC exchange rate sets the undamped natural frequency, and the resistor sets the exponential damping rate.

Standard root form

Write the homogeneous equation as below. Its characteristic equation is $s^2 + 2\alpha s + \omega_0^2 = 0$; comparing coefficients with the circuit ODE identifies α and ω_0 .

$$x'' + 2\alpha x' + \omega_0^2 x = 0$$

Series and parallel parameters

For series RLC, $2\alpha = R/L$; for parallel RLC, $2\alpha = 1/(RC)$. In both ideal topologies $\omega_0 = 1/\sqrt{LC}$. The damping ratio $\zeta = \alpha/\omega_0$ gives a dimensionless classification.

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \alpha_{\text{series}} = \frac{R}{2L}, \quad \alpha_{\text{parallel}} = \frac{1}{2RC}$$

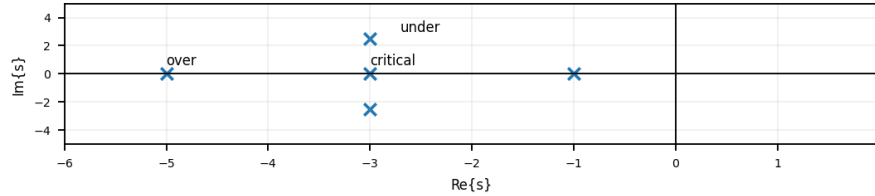
Root location predicts the waveform class

If $\alpha > \omega_0$, the roots are two distinct negative reals (overdamped). If $\alpha = \omega_0$, there is a repeated real root (critically damped). If $\alpha < \omega_0$, the roots are complex conjugates $-\alpha \pm j\omega_d$ with $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ (underdamped).

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}, \quad \zeta = \frac{\alpha}{\omega_0}$$

Deeper engineering point

Because ζ is dimensionless, it lets you compare circuits of very different scale. Two RLC networks with quite different component values but the same ζ have similar normalised transients when time is measured in units of $1/\omega_0$. This is why second-order response charts are parameterised by ζ .



Worked example — classify from component values

A series circuit has $R = 12\ \Omega$, $L = 20\ \text{mH}$ and $C = 100\ \mu\text{F}$.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2 \times 10^{-6}}} = 707.1\ \text{rad/s}, \quad \alpha = \frac{R}{2L} = \frac{12}{0.04} = 300\ \text{s}^{-1}$$

So $\zeta = 0.424 < 1$: underdamped. The damped frequency is $\omega_d = \sqrt{707.1^2 - 300^2} = 640.3\ \text{rad/s}$, and the envelope decays as e^{-300t} .

5. Overdamped response and two real circuit poles

When damping exceeds the LC exchange rate, the natural response is a sum of two decaying exponentials. Both poles lie on the negative real axis, so there is no ringing.

Mathematical form

For $\alpha > \omega_0$ the roots are $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$, both negative in a passive circuit:

$$x_n(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

The dominant slow pole

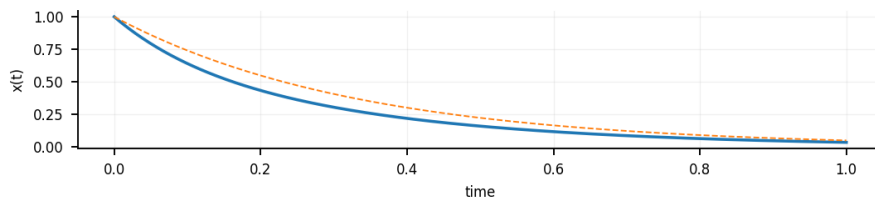
The pole closest to the imaginary axis decays most slowly. At late times the fast mode has largely vanished and the response looks first order, even though two modes were needed to satisfy the initial conditions.

Physical interpretation

Large series resistance dissipates energy before L and C can exchange it repeatedly; in a parallel RLC, small resistance gives strong shunt conductance with the same effect. "Overdamped" describes the root structure, not merely a slow-looking response.

Deeper engineering point

When the poles are widely separated, the fast exponential vanishes so quickly that measurements mainly show the slow pole, giving an effective first-order model after an initial boundary layer. Dominant-pole model reduction is common, but the discarded fast mode can still matter for switching stress and very short-time behaviour.



Worked example — compute both decay modes

For a series RLC with $L = 0.10 \text{ H}$, $C = 100 \mu\text{F}$ and $R = 100 \Omega$: $\omega_0 = 316.2 \text{ rad/s}$ and $\alpha = 500 \text{ s}^{-1}$, so $\sqrt{\alpha^2 - \omega_0^2} = 387.3 \text{ s}^{-1}$ and

$$s_1 = -112.7 \text{ s}^{-1}, \quad s_2 = -887.3 \text{ s}^{-1}$$

The second mode decays almost eight times faster; after a few milliseconds the -112.7 s^{-1} pole dominates the tail. The constants still depend on both initial conditions.

6. Critical damping and repeated poles

Critical damping is the boundary between real-pole and complex-pole behaviour. For the ideal second-order form it gives the fastest return to equilibrium without oscillation.

Repeated-root solution

At $\alpha = \omega_0$ the characteristic equation has the repeated root $s = -\alpha$. Two independent functions are still needed, so the factor t is essential:

$$x_n(t) = (C_1 + C_2 t)e^{-\alpha t}$$

Critical resistance

For a series RLC circuit, $R_{\text{crit}} = 2\sqrt{L/C}$. For a parallel RLC, setting $1/(2RC) = 1/\sqrt{LC}$ gives $R_{\text{crit}} = 1/2\sqrt{L/C}$. These apply only to the canonical topologies; a general network must first be reduced to its actual ODE coefficients.

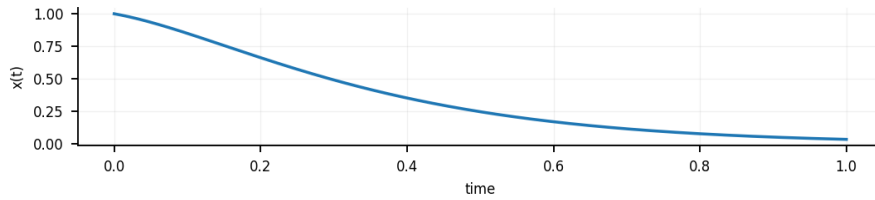
$$R_{\text{crit, series}} = 2\sqrt{\frac{L}{C}}$$

Why "fastest" needs context

Critical damping is fastest among the standard responses without overshoot for a fixed ω_0 . It does not mean that changing parameters always makes a practical circuit faster: changing R can also alter loading, forcing and the measured output.

Deeper engineering point

Critical damping is a boundary, so it is sensitive to tolerances. A resistor chosen exactly from nominal L and C may make a real circuit slightly over- or underdamped. Practical designs usually target a damping margin rather than an exact repeated pole.



Worked example — choose R for critical damping

Let $L = 40 \text{ mH}$ and $C = 10 \text{ }\mu\text{F}$ in a series RLC circuit.

$$R_{\text{crit}} = 2 \sqrt{\frac{L}{C}} = 2 \sqrt{\frac{0.04}{10^{-5}}} = 126.5 \text{ }\Omega$$

Also $\omega_0 = 1/\sqrt{LC} = 1581 \text{ rad/s}$ and $\alpha = R/(2L) = 126.5/0.08 \approx 1581 \text{ s}^{-1}$, confirming $\alpha = \omega_0$. The source-free response is $(C_1 + C_2 t) e^{-1581t}$, with constants set by the initial voltage and current.

7. Underdamped response, damped frequency and ringing

When damping is weaker than the LC exchange rate, the poles are complex conjugates. The response oscillates at a damped frequency inside an exponential envelope that removes energy.

Complex poles become a real sinusoid

For $\alpha < \omega_0$ the roots are $-\alpha \pm j\omega_d$, and the real response can be written in either form below (or as $Me^{-\alpha t} \cos(\omega_d t - \phi)$).

$$s = -\alpha \pm j\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$x_n = e^{-\alpha t} [A \cos \omega_d t + B \sin \omega_d t]$$

Two time scales

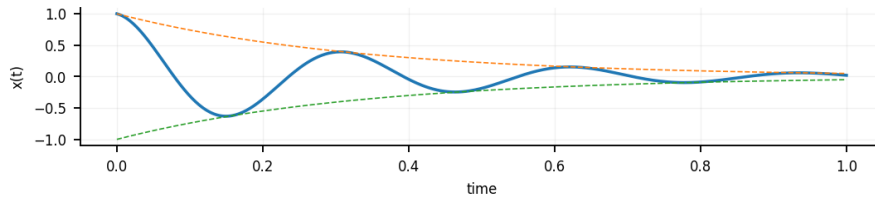
The oscillation period is $T_d = 2\pi/\omega_d$; the envelope time constant is about $1/\alpha$. One sets how fast the cycles occur, the other how quickly their amplitude decays.

Ringing and overshoot

A step response may overshoot, because energy stored in L and C carries the state past equilibrium before dissipation reverses it. The overshoot depends on the damping ratio and on the input/output configuration, so the roots predict whether ringing is possible, not every amplitude detail.

Deeper engineering point

With $\alpha > 0$, the damped frequency ω_d is always below ω_0 . For light damping they are close but not interchangeable: resonant forcing, free ringing and bandwidth calculations involve different characteristic frequencies, so check which one a formula needs.



Worked example — read frequency and envelope from the roots

Suppose the poles are $-200 \pm j600 \text{ s}^{-1}$, so $\alpha = 200 \text{ s}^{-1}$ and $\omega_d = 600 \text{ rad/s}$. The ringing period is $T_d = 2\pi/600 = 10.47 \text{ ms}$, and the envelope falls by e^{-1} every $1/\alpha = 5 \text{ ms}$.

$$\omega_0 = \sqrt{\alpha^2 + \omega_d^2} = 632.5 \text{ rad/s}, \quad \zeta = \frac{\alpha}{\omega_0} = 0.316$$

Any real natural response must have the form $e^{-200t}[A\cos 600t + B\sin 600t]$.

8. Initial conditions from stored capacitor and inductor energy

Second-order constants should be tied to physical stored energy. Capacitor voltage and inductor current are continuous; the derivative of the chosen scalar variable comes from the element laws and the post-switch circuit.

$$v_C(0^+) = v_C(0^-), \quad i_L(0^+) = i_L(0^-)$$

Using v_C as the ODE variable

In a series RLC, $i = C v_C'$. So $v_C(0^+)$ comes from capacitor continuity, and $v_C'(0^+)$ from the continuous inductor (loop) current. Both mathematical initial conditions are then directly physical.

$$v_C'(0^+) = \frac{i(0^+)}{C}$$

Using current as the ODE variable

If the ODE is written for the loop current, continuity gives $i(0^+) = i_L(0^-)$, and the derivative comes from post-switch KVL:

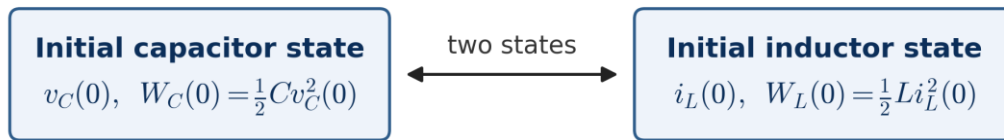
$$i'(0^+) = \frac{v_s(0^+) - Ri(0^+) - v_C(0^+)}{L}$$

Energy is a strong plausibility check

Compute W_C and W_L before switching if useful. In a passive source-free network the total stored energy must not increase; an envelope that grows in a passive circuit signals a sign or root error.

Deeper engineering point

Initial states can be non-zero even when the post-switch source is zero; that is the signature of stored energy. Conversely, a source can switch abruptly while the state stays continuous. Keeping source discontinuity and state continuity separate prevents a common mistake: setting v_C or i_L to zero just because the source changed at $t = 0$.



Worked example — convert physical states into $y(0)$ and $y'(0)$

A source-free series RLC has $C = 50 \mu\text{F}$, $L = 20 \text{ mH}$, $v_C(0) = 4 \text{ V}$ and $i(0) = 0.10 \text{ A}$. Choose $y = v_C$. Then $y(0) = 4 \text{ V}$ and

$$y'(0) = \frac{i(0)}{C} = \frac{0.10}{50 \times 10^{-6}} = 2000 \text{ V/s}$$

These two conditions determine A and B in the natural response. The initial stored energy is $1/2 C (4^2) + 1/2 L (0.1^2) = 0.5 \text{ mJ}$.

9. DC final value, energy decay and interpretation

Long-time behaviour and energy balance give independent checks on an RLC solution. At DC the derivatives vanish; in a source-free passive circuit, resistor dissipation removes stored energy monotonically.

DC equivalent circuits

After the transients of a constant source die away, a capacitor acts as an open circuit and an inductor as a short circuit. Use these replacements to find the final capacitor voltage, inductor current and outputs. A solution approaching any other value is wrong, however consistent its algebra.

Energy balance in a source-free series RLC

With no source, KVL is $Li' + Ri + v_C = 0$. Multiplying by i and using $i = C v_C'$ gives

$$\frac{d}{dt} \left[\frac{1}{2} L i^2 + \frac{1}{2} C v_C^2 \right] = -R i^2 \leq 0$$

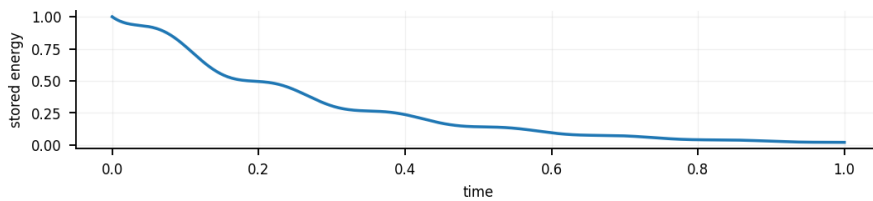
The resistor continuously removes energy; L and C only exchange what remains.

Damping seen through energy

In an underdamped response, energy passes back and forth several times before it is dissipated. Critically damped and overdamped responses dissipate enough that repeated sign reversals do not occur. The pole picture and the energy picture describe the same physics.

Deeper engineering point

The energy derivative is a Lyapunov-style stability argument for passive RLC circuits: if W is positive definite and $dW/dt \leq 0$, stored energy cannot grow without an external supply. This establishes qualitative stability without solving the characteristic equation, and it extends to much larger passive networks.



Worked example — final value plus an energy check

A series RLC is connected to a 10 V DC source. At long times the capacitor is open, so the current is zero, the resistor and inductor drops are zero, and $v_C(\infty) = 10$ V. A computed solution tending to 8 V can be rejected without differentiating anything.

For the source-free version, $W(t)$ must decrease. A candidate natural response containing e^{+50t} would eventually grow in energy, which is impossible for positive R with no active source.

10. Consolidation: classify an RLC response before solving it

Second-order work is much more reliable when classification comes before constant-solving. Establish the topology, coefficients, roots and physical initial and final data before expanding any algebra.

Step 1 — derive the actual ODE

Choose the state or output, write KCL/KVL and substitute the element laws. Normalise to $x'' + 2\alpha x' + \omega_0^2 x = g(t)$. Use the memorised α formulas only if the circuit is exactly the canonical series or parallel topology.

Step 2 — classify the poles

Compare α with ω_0 , or solve the characteristic equation. Write down the response family (two exponentials, repeated-root form or damped sinusoid) before applying initial conditions.

Step 3 — apply the physics and verify

Convert $v_C(0^-)$ and $i_L(0^-)$ into the two mathematical initial conditions. Find the DC final value for constant forcing, determine the constants, and check units, initial derivative, final value, pole stability and energy behaviour.

Deeper engineering point

Pole classification does not depend on initial conditions: the initial state changes modal coefficients, not pole locations. One initial state may make an underdamped circuit ring strongly and another may suppress part of the sinusoid, but neither can turn complex poles into real ones. Keep system properties separate from excitation-dependent amplitudes.

**Worked checkpoint — classify without solving for constants**

A series RLC has $R = 40\ \Omega$, $L = 50\ \text{mH}$ and $C = 200\ \mu\text{F}$. Then $\alpha = R/(2L) = 400\ \text{s}^{-1}$ and $\omega_0 = 1/\sqrt{LC} = 316.2\ \text{rad/s}$, so $\alpha > \omega_0$: overdamped.

$$s = -400 \pm \sqrt{400^2 - 316.2^2} = -155.1 \text{ and } -644.9\ \text{s}^{-1}$$

Before using any initial conditions we know the natural response is $C_1 e^{-155.1t} + C_2 e^{-644.9t}$, with the slower pole controlling the late tail.