

From Circuit Laws to First-Order ODE Models

Ordinary Differential Equations for Electrical Engineers — Ch. 6 of 15

1. Lumped circuits, KCL, KVL and sign conventions

Circuit ODEs begin with physical laws, not a memorised template. First decide whether a lumped model is appropriate, assign reference directions consistently, and then write Kirchhoff equations that respect those choices.

The lumped approximation

A lumped circuit treats each component as concentrated at a point, assuming electrical dimensions are small compared with the wavelength and propagation distances. Voltage is then well defined between nodes, and the current in a branch is a single function of time. This is excellent for many low-frequency, moderate-size circuits, but it is not a universal law: long interconnects and high-frequency structures eventually need distributed transmission-line or field models.

$$\sum_k i_k(t) = 0 \quad \text{at an ideal node}$$

KCL and KVL are conservation statements

Kirchhoff's current law (KCL) expresses charge conservation at a node, neglecting charge storage in the ideal node itself. Kirchhoff's voltage law (KVL) states that branch voltages around an ideal lumped loop sum to zero. Choose current arrows and voltage polarities before writing equations; a negative answer later simply means the true direction is opposite to the reference.

$$\sum_k v_k(t) = 0 \quad \text{around an ideal loop}$$

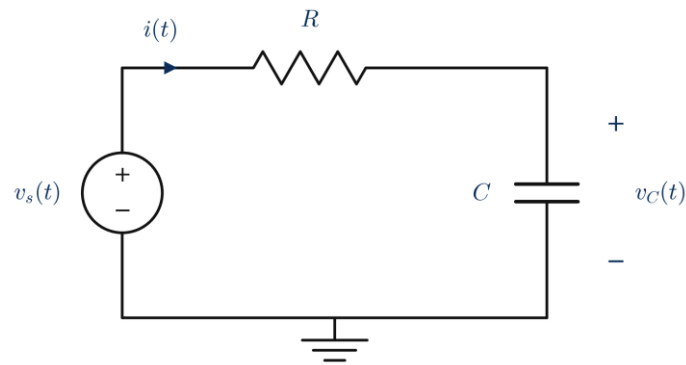
Passive sign convention

Under the passive sign convention, current enters the terminal marked positive. Then $p(t) = v(t) i(t) > 0$ means the element absorbs power, and $p(t) < 0$ means it delivers power. This is especially useful when checking resistor loss and stored energy after an ODE has been derived.

$$p(t) = v(t) i(t) \quad (\text{passive sign convention})$$

Deeper engineering point

Kirchhoff's laws are exact conservation statements only within the lumped model. At high enough frequency, conductors acquire distributed voltage and phase, and a single KVL loop may no longer represent the field correctly. The modelling skill has two parts: apply KCL/KVL consistently, and know when the circuit abstraction has reached its physical limit.



Series RC circuit with loop current $i(t)$ and capacitor voltage $v_C(t)$.

Worked example — write the equation before solving it

A source $v_s(t)$ drives a series resistor R and capacitor C . Choose the loop current $i(t)$ entering the positive terminal of the capacitor. KVL gives

$$v_s = v_R + v_C = Ri + v_C$$

Using $i = C dv_C/dt$ gives the first-order model

$$RC \frac{dv_C}{dt} + v_C = v_s(t)$$

The model has not been solved yet; it has been derived from conservation plus the capacitor law. Separating modelling from solution is a key engineering habit.

2. Resistor, capacitor and inductor element laws

KCL and KVL say how branches connect; constitutive (element) relations say how each element behaves. For ideal R, C and L these relations decide whether the circuit equation is algebraic or differential.

Resistor: instantaneous dissipation

For an ideal resistor, voltage and current are related algebraically: there is no derivative or integral, so an ideal resistor has no state or memory. With the passive sign convention, $p_R = Ri^2 = v^2/R \geq 0$: electrical energy is irreversibly converted to heat.

$$v_R = R i_R$$

Capacitor: voltage is the state

A capacitor stores charge $q = Cv_C$; differentiating gives $i_C = C dv_C/dt$. A finite current produces a finite slope, so the capacitor voltage cannot jump unless an impulse current flows. Choosing capacitor voltage as the unknown therefore introduces a first derivative.

$$i_C = C \frac{dv_C}{dt}, \quad q = Cv_C$$

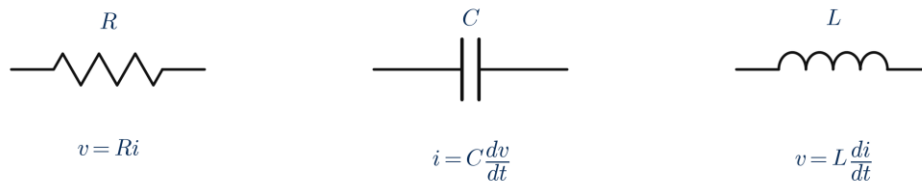
Inductor: current is the state

For an ideal linear inductor the flux linkage is $\lambda = Li_L$ and $v_L = L di_L/dt$. A finite voltage produces a finite current slope, so inductor current cannot jump without an impulse voltage. Capacitor and inductor are duals: C stores electric-field energy through voltage, L stores magnetic-field energy through current.

$$v_L = L \frac{di_L}{dt}, \quad \lambda = Li_L$$

Deeper engineering point

These ideal laws assume linear, time-invariant components with constant R , C and L . Real capacitors have leakage and series resistance; real inductors have winding resistance, parasitic capacitance and saturation. Such effects can be added as extra elements or nonlinear laws. The ODE method stays the same, but the order and linearity may change.



Element symbols and their constitutive laws.

Worked example — translate a waveform through an element law

Let $v_C(t) = 6e^{-t/0.020}$ V for $t \geq 0$ and $C = 100 \mu\text{F}$. Then

$$\frac{dv_C}{dt} = -\frac{6}{0.020} e^{-t/0.020} = -300 e^{-50t} \text{ V/s}$$

so

$$i_C(t) = C \frac{dv_C}{dt} = -0.030 e^{-50t} \text{ A}$$

The negative sign is not an error: with the chosen passive reference, the capacitor is discharging and delivering energy to the rest of the circuit.

3. Energy storage and why C and L create memory

An ODE is needed when the present rate of change depends on energy already stored in the system. Capacitors and inductors create this memory; resistors dissipate energy but store no independent state.

Energy in a capacitor

Starting from $p = vi$ and $i = C dv/dt$ and integrating power over time, the stored electric energy is $1/2 C v_C^2$. Because it depends only on v_C , knowing the capacitor voltage at the switching instant is equivalent to knowing its stored energy.

$$W_C = \frac{1}{2} C v_C^2$$

Energy in an inductor

Using $p = vi$ and $v = L di/dt$ gives $1/2 Li_L^2$, so inductor current carries the magnetic-energy state. One independent capacitor or inductor usually gives a first-order ODE; two independent storage states can produce second-order dynamics.

$$W_L = \frac{1}{2} Li_L^2$$

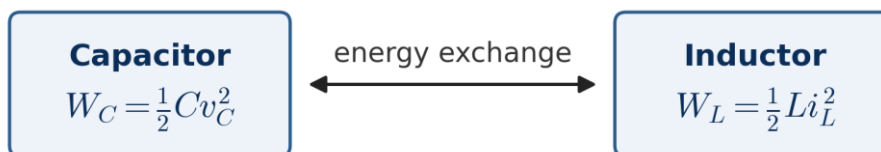
Why state continuity matters

If v_C changed by a finite amount in zero time, dv_C/dt would contain an impulse and i_C would be impulsive; likewise, a jump in i_L needs an impulsive inductor voltage. Circuits driven by finite sources therefore satisfy the continuity conditions below. They are often more useful than a memorised response formula.

$$v_C(0^+) = v_C(0^-), \quad i_L(0^+) = i_L(0^-) \quad (\text{finite excitation})$$

Deeper engineering point

State and energy are related but not identical. A state variable is chosen to predict future motion; energy is a physical scalar used for interpretation and passivity checks. For a linear capacitor or inductor the natural state fixes the stored energy directly; in larger networks, several state coordinates can represent the same total energy through a quadratic form.



Worked example — how much "memory" is stored?

A 220 μF capacitor charged to 12 V stores

$$W_C = \frac{1}{2} (220 \times 10^{-6}) (12^2) = 15.84 \text{ mJ}$$

If it is suddenly connected to a resistor, its voltage just after switching is still 12 V: the resistor changes the subsequent slope, not the instantaneous state. For comparison, a 50 mH inductor carrying 0.8 A stores $1/2 (0.05) (0.8^2) = 16 \text{ mJ}$. Similar energies, different state variables.

4. Deriving the series RC circuit ODE with KVL

The series RC circuit is the canonical first-order voltage model. Its ODE follows from one KVL equation and the capacitor law, and the time constant appears naturally on normalising.

Choose the capacitor voltage as the state

With a source $v_s(t)$, resistor R and capacitor C in series, the same current flows through both elements. Taking $v_C(t)$ as the unknown avoids an integral, because $i = C dv_C/dt$ substitutes directly into KVL.

$$v_s = Ri + v_C$$

Derive and normalise

Substituting $i = C dv_C/dt$ gives the model below. Dividing by RC exposes the decay rate $1/(RC)$; since RC has units of seconds, $\tau = RC$ is the natural time scale.

$$RC \frac{dv_C}{dt} + v_C = v_s(t)$$

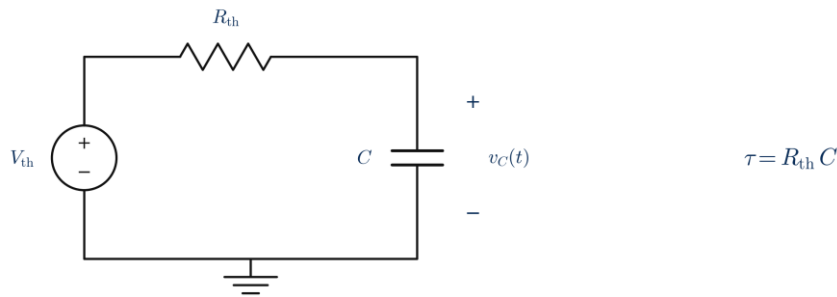
Read the equation physically

The derivative term represents the capacitor's resistance to rapid voltage change; the v_C term comes from the resistor current after substitution; the source is the forcing. For a constant source V_s with positive R and C , the derivative vanishes at long times and $v_C \rightarrow V_s$.

$$\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{1}{RC} v_s(t), \quad \tau = RC$$

Deeper engineering point

The derivation generalises. Seen from the capacitor terminals, any linear resistive, source-driven network with one independent capacitor can be replaced by a Thévenin source $V_{th}(t)$ and resistance R_{th} . The capacitor then satisfies $R_{th}C v_C' + v_C = V_{th}$, so the first-order structure is a network property, not a special-case formula.



Thévenin equivalent seen by the capacitor.

Worked example — non-zero initial capacitor voltage

Let $R = 2.0 \text{ k}\Omega$, $C = 50 \text{ }\mu\text{F}$, $v_s = 10 \text{ V}$ for $t \geq 0$, and $v_C(0) = 2 \text{ V}$. Then $\tau = RC = 0.10 \text{ s}$ and the final value is 10 V :

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)]e^{-t/\tau} = 10 - 8e^{-10t} \text{ V}$$

Check the initial slope: from the ODE, $v_C'(0) = (10 - 2)/0.10 = 80 \text{ V/s}$; differentiating the solution gives the same value.

5. Deriving the parallel RC circuit ODE with KCL

For a parallel RC network, the natural equation comes from KCL at the common node. The node voltage is also the capacitor voltage, so the model is very direct.

Write KCL at the node

A current source $i_s(t)$ feeds a resistor R and capacitor C in parallel. With node voltage $v(t)$, the resistor current is v/R and the capacitor current is $C dv/dt$, and KCL says the source current equals their sum.

$$i_s = i_R + i_C = \frac{v}{R} + C \frac{dv}{dt}$$

Normalised form and time constant

Dividing by C gives the normalised form. The same $\tau = RC$ appears as in the series circuit, because the natural response involves the same resistor–capacitor exchange; only the forcing term and the most convenient circuit law change.

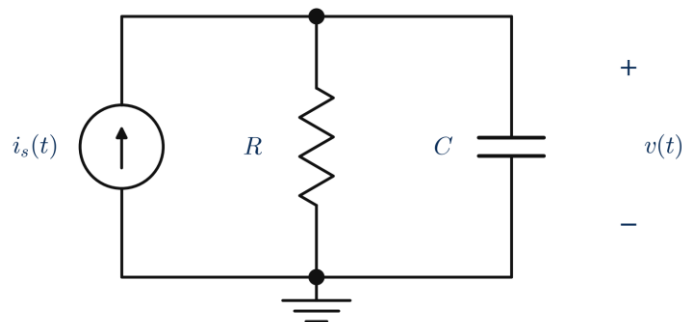
$$\frac{dv}{dt} + \frac{1}{RC} v = \frac{i_s(t)}{C}$$

Final value under DC current forcing

At DC the capacitor is an open circuit, so all the source current flows through R and $v(\infty) = RI_s$ for a constant source I_s . This is an immediate check on any solution.

Deeper engineering point

The parallel RC equation is the Norton dual of the series RC equation. A current source with shunt resistance leads naturally to KCL; a voltage source with series resistance leads to KVL. Converting between Thévenin and Norton forms changes how the forcing is written, not the physical response.



Parallel RC circuit driven by a current source.

Worked example — current-source charging

A 3 mA source is switched onto $R = 1.0 \text{ k}\Omega$ in parallel with $C = 100 \text{ }\mu\text{F}$, with $v(0) = 0$. Then $\tau = RC = 0.10 \text{ s}$ and the DC final value is $RI_s = 3 \text{ V}$:

$$v(t) = 3[1 - e^{-t/0.10}] \text{ V}$$

At $t = 0^+$ the resistor carries no current because $v = 0$, so all 3 mA enters the capacitor: $v'(0) = i/C = 30 \text{ V/s}$, matching the derivative of the solution.

6. Deriving the series and parallel RL circuit ODEs

RL circuits are the magnetic dual of RC circuits. The inductor current is the natural state, and the time constant is L divided by the resistance seen by the inductor.

Series RL from KVL

For a source $v_s(t)$ driving R and L in series, KVL gives $v_s = Ri + L di/dt$. Normalising gives $i' + (R/L)i = v_s/L$; the coefficient R/L has units s^{-1} , so $\tau = L/R$.

$$L \frac{di}{dt} + Ri = v_s(t) \quad (\text{series RL})$$

Parallel RL from KCL

For a current source i_s feeding R and L in parallel, the node voltage is $v = L di_L/dt$ and the resistor current is v/R . KCL gives $i_s = i_L + (L/R) i_L'$; multiplying by R/L gives $i_L' + (R/L) i_L = (R/L) i_s$. The same natural rate R/L appears.

$$\frac{L}{R} \frac{di_L}{dt} + i_L = i_s(t) \quad (\text{parallel RL})$$

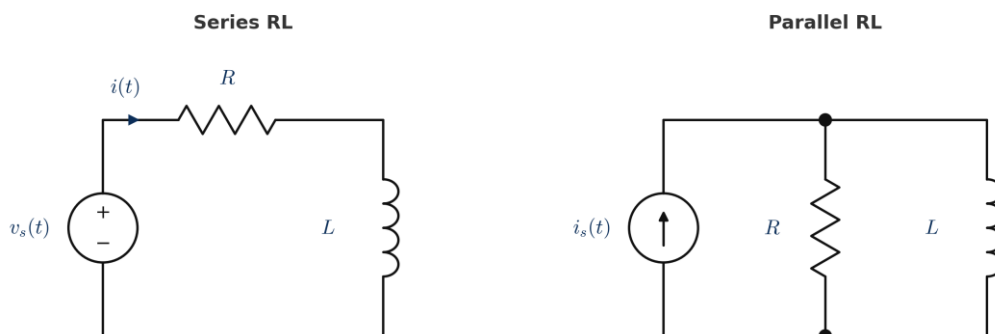
Initial and final behaviour

Inductor current is continuous for finite voltage, and under a DC source an ideal inductor behaves as a short circuit once transients have died away. These two facts usually give $i(0^+)$ and $i(\infty)$ before the ODE is solved.

$$\tau = \frac{L}{R}$$

Deeper engineering point

For a general first-order RL network, the relevant resistance is that seen from the inductor terminals with independent sources suppressed. This is why $\tau = L/R_{eq}$ works even when the inductor is not visibly in series with a single resistor: the formula comes from the source-free coefficient of i_L , not from how the diagram looks.



Series RL (voltage-driven) and parallel RL (current-driven) circuits.

Worked example — current build-up in a series RL circuit

Let $R = 20\ \Omega$, $L = 0.10\ \text{H}$ and $v_s = 5\ \text{V}$ be applied at $t = 0$ with $i(0) = 0$. Then $\tau = L/R = 5\ \text{ms}$ and the DC current is $V_s/R = 0.25\ \text{A}$:

$$i(t) = 0.25[1 - e^{-t/0.005}] \text{ A}$$

The current cannot jump, but its slope can: $i'(0) = v_s/L = 50\ \text{A/s}$ because the resistor drop is initially zero. At long times $i' \rightarrow 0$ and the resistor drop tends to $5\ \text{V}$.

7. Initial conditions: continuity of v_C and i_L

Initial conditions in circuit ODEs are physical states inherited from the circuit just before switching. Find them from pre-switch behaviour and continuity before solving any post-switch equation.

Use 0^- and 0^+ deliberately

0^- is the instant just before switching and 0^+ the instant just after. For finite excitation the capacitor voltage and inductor current are continuous. Resistor voltage and current can change instantly, because a resistor stores no energy.

$$v_C(0^+) = v_C(0^-)$$

Pre-switch steady state is often simple

If a circuit has been connected to DC for a long time before $t = 0$, capacitors act as open circuits and inductors as short circuits. Ordinary DC analysis then gives the pre-switch states, which become the initial conditions of the post-switch ODE.

$$i_L(0^+) = i_L(0^-)$$

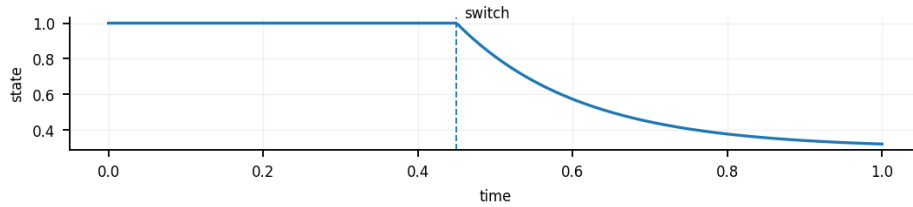
Initial derivatives come from the new topology

Continuity supplies the state, not usually its derivative. To find $v_C'(0^+)$ or $i_L'(0^+)$, apply KCL/KVL to the post-switch circuit using the known state value. This is a powerful check on the sign and early direction of the solution.

$$v_C'(0^+) = \frac{i_C(0^+)}{C}, \quad i_L'(0^+) = \frac{v_L(0^+)}{L}$$

Deeper engineering point

Continuity follows from excluding impulses; it is not an absolute law of nature. An ideal impulse current can change the capacitor charge instantly, giving $\Delta v_C = \frac{1}{C} \int i\ dt$; an impulse voltage can likewise change the inductor current. These cases matter later, when impulse inputs and Laplace transforms are introduced.



Worked example — the first post-switch slope

A capacitor has $v_C(0^-) = 8 \text{ V}$. At $t = 0$ it is connected through $R = 2 \text{ k}\Omega$ to a 2 V source, with $C = 100 \text{ }\mu\text{F}$. Continuity gives $v_C(0^+) = 8 \text{ V}$. The post-switch ODE is $RC v_C' + v_C = 2$, so

$$v_C'(0^+) = \frac{2 - 8}{RC} = \frac{-6}{0.20} = -30 \text{ V/s}$$

The negative slope is physically correct: the capacitor starts above its new 2 V equilibrium and must discharge towards it.

8. Time constants, source-free response and DC final conditions

A first-order circuit can usually be characterised before detailed algebra by three quantities: the time constant, the state just after switching and the final DC state. Together they determine the whole exponential response under constant forcing.

Source-free natural response

With independent sources set to zero, a first-order state x satisfies $x' + x/\tau = 0$, so $x(t) = x(0) e^{-t/\tau}$. It falls to $e^{-1} \approx 36.8\%$ of its initial value after one time constant and below 1% after about five.

$$x_n(t) = K e^{-t/\tau}$$

Resistance seen by the storage element

For a linear resistive network, $\tau = R_{\text{eq}}C$ for a capacitor and $\tau = L/R_{\text{eq}}$ for an inductor, where R_{eq} is the resistance seen from the storage element with independent voltage sources shorted and independent current sources opened. Dependent sources stay active and may need a test source.

$$\tau_{RC} = R_{\text{eq}} C, \quad \tau_{RL} = \frac{L}{R_{\text{eq}}}$$

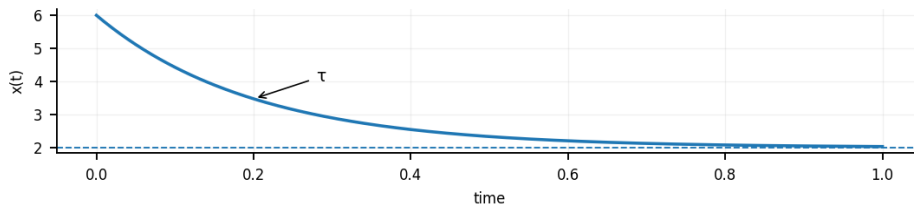
DC final conditions

For constant sources after switching, set the derivatives to zero; equivalently, treat the capacitor as open and the inductor as short. Solving the resulting resistive circuit gives $x(\infty)$, and the transient bridges continuously from $x(0^+)$ to $x(\infty)$.

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau}$$

Deeper engineering point

Equivalent-resistance shortcuts need care with dependent sources. Short independent voltage sources and open independent current sources, but leave controlled sources active. Then apply a test voltage or current at the storage-element port and compute $R_{eq} = V_{test}/I_{test}$. This is the rigorous route to the time constant of an active network.



Worked example — use only the initial value, final value and τ

A $20\ \mu\text{F}$ capacitor sees $R_{eq} = 5\ \text{k}\Omega$ after switching, so $\tau = 0.10\ \text{s}$. Its voltage moves from $v_C(0^+) = 6\ \text{V}$ towards a DC final value of $2\ \text{V}$. Without re-deriving the ODE:

$$v_C(t) = 2 + (6 - 2)e^{-t/0.10} = 2 + 4e^{-10t}\ \text{V}$$

At $t = 0.10\ \text{s}$ the transient part is $4/e = 1.472\ \text{V}$, so $v_C \approx 3.472\ \text{V}$. After five time constants the voltage is within about $0.027\ \text{V}$ of $2\ \text{V}$.

9. Complete response and verification by units and limits

A correct first-order solution must satisfy the differential equation, the initial state, the final physical behaviour and dimensional consistency. Verification is part of the solution, not an optional extra.

Natural plus forced response

For linear circuits the complete response is the natural response plus a particular (forced) response. Under constant forcing the particular solution is the DC final value, so $x = x(\infty) + Ke^{-t/\tau}$, and continuity at the switching instant fixes K .

$$x(t) = x_f + (x_0 - x_f)e^{-t/\tau}$$

Four fast checks

- Units:** terms added in the ODE have the same dimensions, and t/τ is dimensionless.
- Initial value:** setting $t = 0$ recovers the initial condition.
- Final value:** letting $t \rightarrow \infty$ agrees with the DC equivalent circuit.
- Substitution:** differentiating and substituting back into the ODE catches sign and time-constant errors.

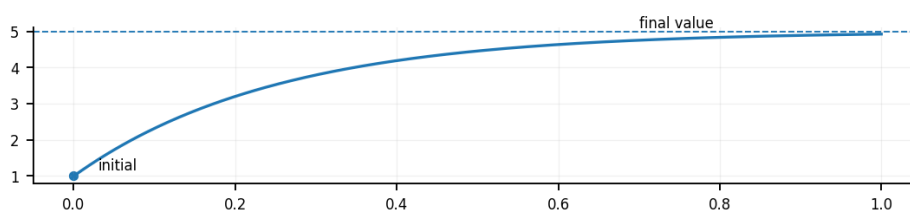
$$x(0) = x_0, \quad \lim_{t \rightarrow \infty} x(t) = x_f$$

Energy and monotonicity

In a passive, source-free first-order RC or RL network, stored energy must decrease; a magnitude that grows without an active source is a warning sign. With constant forcing the state may rise or fall, but it must move towards the equilibrium predicted by the resistive DC circuit.

Deeper engineering point

A particularly strong check is to compare the initial slope of the closed-form response with the slope implied directly by the circuit at the switching instant. The initial and final values can both be right while the time constant has the wrong sign or size; the initial-slope check often exposes that error immediately.



Worked example — verify rather than trust the formula

For $RC\ v' + v = V_s$ let $R = 1\text{ k}\Omega$, $C = 1\text{ mF}$, $V_s = 5\text{ V}$ and $v(0) = 1\text{ V}$. Proposed solution: $v = 5 - 4e^{-t}$.

Units: $RC = 1\text{ s}$, so the exponent is dimensionless. **Initial value:** $v(0) = 1\text{ V}$. **Final value:** $v \rightarrow 5\text{ V}$.

Substitution: $v' = 4e^{-t}$, so $v' + v = 4e^{-t} + 5 - 4e^{-t} = 5$, exactly the normalised ODE. All checks agree, so the solution is internally and physically consistent.

10. Consolidation: a repeatable recipe for first-order circuit ODEs

A reliable modelling workflow reduces sign errors and stops solution methods being applied to the wrong equation. The same sequence works for RC, RL and many more complicated networks.

Step 1 — choose the state and the references

Mark voltage polarities and current arrows. Prefer capacitor voltage or inductor current as the state, since their element laws contain a first derivative directly. Record the known pre-switch conditions.

Step 2 — write conservation laws before substituting

Use one independent KCL or KVL equation that contains the state branch. Only then insert $v_R = Ri$, $i_C = Cv_C'$ or $v_L = Li_L'$. Rearrange into standard first-order form and check the units.

Step 3 — predict before calculating constants

Find $x(0^+)$, $x(\infty)$ and τ ; they predict the direction and speed of the response. Solve for the constant, then verify the ODE, continuity and limiting behaviour. If a check fails, return to the modelling stage rather than patching the final algebra.

$$x(t) = x(\infty) + [x(0^+) - x(\infty)]e^{-t/\tau} \quad (\text{constant forcing})$$

Deeper engineering point

Keep modelling and solving separate. Two correct solution methods applied to a correctly derived ODE give equivalent answers, but no method can rescue an ODE built on the wrong topology or sign convention. The highest-value reasoning often happens before the integration begins.

**Worked checkpoint — derive an RL model in five lines**

A 12 V source drives $R = 6\ \Omega$ and $L = 30\ \text{mH}$ in series, starting from zero current.

1. KVL: $12 = 6i + 0.03\ i'$.
2. Normalise: $i' + 200\ i = 400$.
3. Time constant: $\tau = L/R = 5\ \text{ms}$.
4. Limits: $i(0^+) = 0$ and $i(\infty) = 12/6 = 2\ \text{A}$.
5. Solution: $i(t) = 2(1 - e^{-200t})\ \text{A}$.

Check: $i'(0) = 400\ \text{A/s}$, which also follows directly from $v_L(0) = 12\ \text{V}$ and $i' = v_L/L$.