

Second-Order Linear ODEs and Natural Modes

Ordinary Differential Equations for Electrical Engineers — Foundation Ch. 5 of 15

1. Second-order form and two initial conditions

A second-order ODE contains a second derivative and normally needs two independent initial conditions. In EE these usually represent two independent stored-energy states, or one state and its initial rate of change.

$$a_2 y'' + a_1 y' + a_0 y = f(t), \quad a_2 \neq 0$$

Two modes need two initial conditions

Solving a second-order equation introduces two constants. Physically, second-order circuit models usually contain two storage states, such as a capacitor voltage and an inductor current. A typical IVP supplies $y(t_0)$ and $y'(t_0)$, or circuit quantities that determine them, and both are applied to the complete response. One condition fixes the starting position on the response curve; the other fixes its initial direction.

The two conditions must be independent: giving $y(0)$ twice in different notation adds nothing. When the chosen ODE variable is not a natural state, use element laws and KCL/KVL to convert $v_C(0)$ and $i_L(0)$ into $y(0)$ and $y'(0)$.

Worked example — why two constants appear

Integrate $y'' = 0$ twice:

$$y' = C_1, \quad y = C_1 t + C_2$$

Result: two integrations give two constants, so two independent conditions are needed for a unique solution. Each condition gives one linear equation in the constants; using only one leaves an infinite family of responses.

Typical condition pairs

- $y(0)$ and $y'(0)$ in a mathematical IVP.
- $v_C(0)$ and $i_L(0)$ in a circuit with two storage elements.
- Position and velocity in a mechanical analogue.

Order and state: The order of a minimal ODE equals the number of independent state variables needed to describe the system.

2. Characteristic equation and natural modes

For a homogeneous constant-coefficient equation, try $y = e^{rt}$. Every derivative reproduces the exponential, so the ODE reduces to a polynomial in r .

$$a_2 y'' + a_1 y' + a_0 y = 0, \quad y = e^{rt}$$

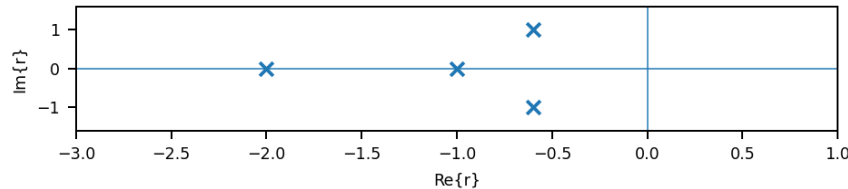
$$a_2 r^2 + a_1 r + a_0 = 0$$

Differentiation becomes algebra

Each derivative contributes a factor r ; the common exponential (never zero) cancels, leaving the characteristic polynomial. Its roots are the natural modes, and their number and type fix the qualitative response before any initial condition is applied. This is the central organising idea for second-order ODEs.

Each distinct root r_k contributes a mode $e^{r_k t}$.

Multiplying the ODE by a non-zero constant does not change the roots, so normalise first if it helps. Check the roots by direct substitution, and by their sum ($-a_1/a_2$) and product (a_0/a_2).



Worked example — derive the modes

Solve $y'' + 5y' + 6y = 0$.

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0 \quad \Rightarrow \quad r_1 = -2, \quad r_2 = -3$$

$$y_h = Ae^{-2t} + Be^{-3t}$$

Result: each characteristic root contributes one independent natural mode.

Key idea: the ODE determines the roots; the initial conditions determine the constants A and B .

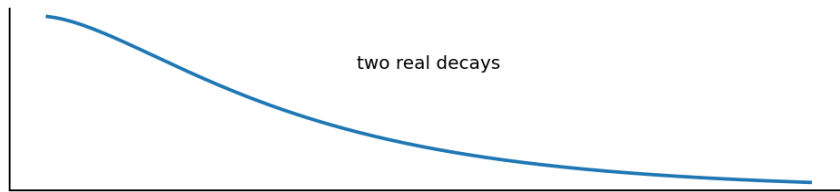
3. Two distinct real roots: overdamped form

With two distinct negative real roots the natural response is a sum of two decaying exponentials. It does not oscillate, and one mode usually disappears sooner than the other.

$$y_h = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 \neq r_2$$

Two decay rates, no oscillation

If both roots are negative, both modes decay. The root closest to zero is the slow mode and dominates the late-time tail; the more negative root dies away faster. The response can still rise before falling if the constants have opposite signs, but real roots never produce sinusoidal oscillation. "Overdamped" does not mean "no transient": it means a weighted sum of real exponentials. For settling time, look at the root nearest the imaginary axis.



Worked example — apply two initial conditions

Solve $y'' + 5y' + 6y = 0$ with $y(0) = 4$ and $y'(0) = -9$. From $y = Ae^{-2t} + Be^{-3t}$:

$$A + B = 4, \quad -2A - 3B = -9 \quad \Rightarrow \quad A = 3, \quad B = 1$$

$$y(t) = 3e^{-2t} + e^{-3t}$$

Result: both modes decay; the e^{-3t} term vanishes faster.

Dominant pole: at long times the root closest to the imaginary axis, here $r = -2$, dominates.

Check: evaluate $y(0)$ and $y'(0)$ explicitly. A plausible-looking decay is not enough if the derivative condition fails.

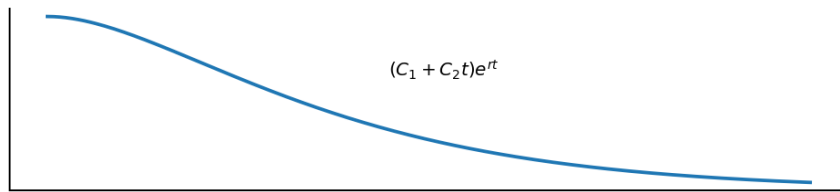
4. Repeated real root: critically damped form

If the characteristic polynomial has a repeated root r , one exponential cannot supply two independent solutions. The second independent solution is te^{rt} .

$$y_h = (A + Bt)e^{rt}$$

A repeated root needs a second shape

Writing $C_1e^{rt} + C_2e^{rt}$ would repeat the same function, leaving only one independent constant; the second solution must be te^{rt} . In the standard second-order model this is critical damping: the boundary between two distinct real roots and a complex-conjugate pair, where the two real roots have merged. The factor t can make the response rise at first, but for $r < 0$ the exponential envelope still controls the long-term behaviour.



Worked example — repeated root

Solve $y'' + 4y' + 4y = 0$ with $y(0) = 2$ and $y'(0) = 0$.

$$(r + 2)^2 = 0 \Rightarrow r = -2, \quad y = (A + Bt)e^{-2t}$$

From $y(0) = 2$, $A = 2$. Differentiating:

$$y' = [B - 2A - 2Bt]e^{-2t}$$

From $y'(0) = 0$, $B - 4 = 0$, so $B = 4$:

$$y = (2 + 4t)e^{-2t}$$

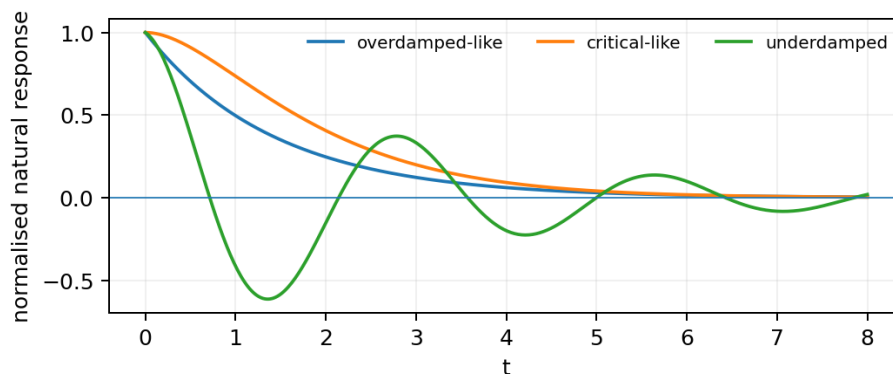
Result: the response is non-oscillatory and sits on the boundary between real-root and complex-root behaviour. Because e^{rt} and $t e^{rt}$ have different initial slopes, the two conditions can fix the two constants.

5. Complex-conjugate roots: damped oscillation

For roots $r = -\alpha \pm j\omega_d$, the real response is a decaying exponential envelope multiplying a sinusoid. This is the mathematical origin of ringing in underdamped circuits.

$$r = -\alpha \pm j\omega_d$$

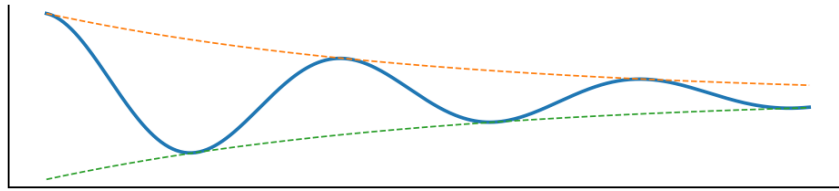
$$y_h = e^{-\alpha t}(A \cos \omega_d t + B \sin \omega_d t)$$



Complex roots separate decay from oscillation

The conjugate exponentials combine into a real damped sinusoid. The real part sets the envelope $e^{-\alpha t}$, with decay time scale about $1/\alpha$; the imaginary part is the damped angular frequency, with period $2\pi/\omega_d$. If $\alpha > 0$ the oscillation shrinks; if $\alpha = 0$ the ideal response is undamped. The initial conditions fix the cosine-sine mixture, equivalently the amplitude and phase.

$$y_h = e^{-\alpha t}[C_1 \cos \omega_d t + C_2 \sin \omega_d t] = A e^{-\alpha t} \cos(\omega_d t - \phi)$$



Worked example — damped oscillation from roots

A system has roots $-3 \pm j4$ and initial conditions $y(0) = 2$, $y'(0) = -6$.

$$y = e^{-3t}(A\cos 4t + B\sin 4t)$$

At $t = 0$, $A = 2$. Differentiating gives $y'(0) = -3A + 4B$, so

$$-6 = -6 + 4B \Rightarrow B = 0, \quad y = 2e^{-3t}\cos 4t$$

Result: envelope decay rate 3 s^{-1} , damped angular frequency 4 rad/s . The envelope should match the real part of the roots and the zero-crossing spacing should match the imaginary part.

6. Undamped oscillation, natural frequency and phase

Without damping, the roots lie on the imaginary axis at $\pm j\omega_0$ and the ideal response oscillates indefinitely. Real passive circuits always have some loss, but the undamped case defines the natural-frequency scale.

$$y'' + \omega_0^2 y = 0$$

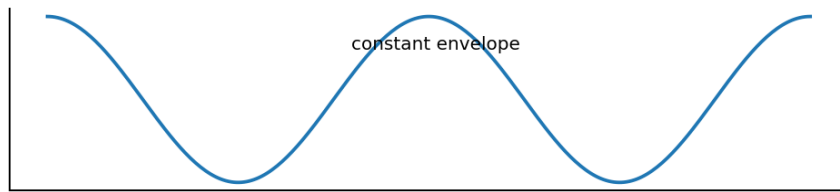
$$y = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \phi)$$

Stored energy exchange

With zero damping the solution is sinusoidal with constant amplitude. In an LC circuit, energy moves back and forth between the capacitor's electric field and the inductor's magnetic field; with no resistance, the total stays constant. The natural frequency is set by the storage elements:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The lack of decay follows from zero resistive loss. Real circuits are always damped, but the ideal model still identifies the natural frequency and the energy exchange at that frequency.



Worked example — LC natural oscillation

For $y'' + 10\,000y = 0$ with $y(0) = 3$ and $y'(0) = 0$, find the response and period.

$$\omega_0 = \sqrt{10\,000} = 100 \text{ rad/s}, \quad y = 3\cos 100t, \quad T = \frac{2\pi}{100} = 62.8 \text{ ms}$$

Result: the ideal energy exchange repeats every 62.8 ms.

Physical interpretation: check that $1/\sqrt{LC}$ has units s^{-1} . The initial capacitor voltage and inductor current set the phase, and voltage and current are a quarter-cycle apart. Resistance introduces damping by removing energy.

7. Particular solutions for common forcing functions

For a non-homogeneous equation, a particular solution accounts for the forcing. With constant coefficients, the method of undetermined coefficients handles polynomial, exponential and sinusoidal inputs well.

Forcing	Initial trial
constant K	A
e^{at}	Ae^{at}
$\cos \omega t$	$A\cos \omega t + B\sin \omega t$
polynomial	polynomial of the same degree

Mirror the forcing

Constants suggest a constant trial; exponentials suggest the same exponential; sines and cosines need both terms, because differentiation rotates between them. First compare the trial with the natural modes: if it duplicates a homogeneous solution, multiply by t until it is linearly independent. The particular coefficients come only from matching the forcing; the initial conditions are used afterwards on $y_h + y_p$.

Worked example — constant forcing

Find a particular solution of $y'' + 4y' + 5y = 10$. Try $y_p = A$, so $y_p' = y_p'' = 0$:

$$5A = 10 \Rightarrow A = 2$$

Result: $y_p = 2$. With stable homogeneous roots, this is also the final value of the complete response.

When the trial overlaps a natural mode

If the trial repeats a term already in y_h , multiply it by t until it is independent. For sinusoidal forcing at a natural frequency, this is resonance.

8. Complete response = natural + forced response

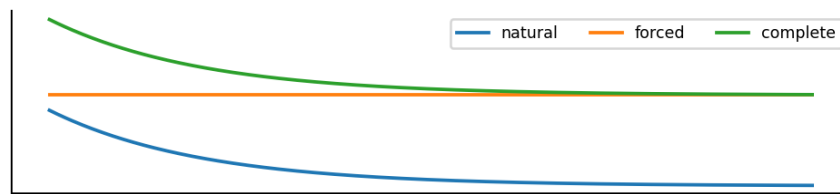
The complete solution of a linear forced second-order ODE is the sum of the homogeneous natural response and one particular forced response. Initial conditions are applied to the sum.

$$y(t) = y_h(t) + y_p(t)$$

Two parts, two physical questions

The natural response depends on the system coefficients and the initial stored energy; the forced response depends on the input. Linearity is what allows them to be added. In a stable damped system $y_h \rightarrow 0$, so the long-term response approaches the particular steady-state solution. If the forcing tends to a constant, set the derivatives to zero in the ODE to find the final value directly.

For stable roots, $\lim_{t \rightarrow \infty} y_h(t) = 0$.



Worked example — constant (step-like) forcing

Solve $y'' + 3y' + 2y = 8$ with $y(0) = 0$ and $y'(0) = 0$.

$$r^2 + 3r + 2 = (r + 1)(r + 2) \Rightarrow y_h = Ae^{-t} + Be^{-2t}$$

For constant forcing $y_p = 4$, so $y = 4 + Ae^{-t} + Be^{-2t}$. The initial conditions give $A = -8$ and $B = 4$:

$$y(t) = 4 - 8e^{-t} + 4e^{-2t}$$

Result: the steady value is 4; the two exponentials form the transient.

Limit check: for a stable system under constant forcing, the natural terms vanish as $t \rightarrow \infty$. Writing $z = y - y_p$ can simplify the algebra: z obeys the homogeneous equation and starts from the initial error.

9. Resonance and repeated forcing roots

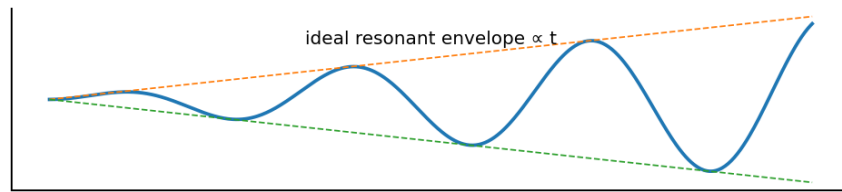
Resonance occurs mathematically when the forcing has the same form as a natural mode. The ordinary trial fails because it is already a homogeneous solution, so it must be multiplied by t . In an ideal undamped oscillator this produces a growing oscillation.

$$y'' + \omega_0^2 y = F \cos \omega_0 t$$

Why the amplitude grows

If the forcing matches a homogeneous solution, the usual trial is annihilated by the differential operator; multiplying by t creates an independent trial. Driven exactly at its natural frequency, an ideal undamped oscillator develops a term whose envelope grows in proportion to t . Real circuits have damping and nonlinear limits, so physical amplitudes stay finite, and the peak frequency need not equal the undamped natural frequency. The ideal case still explains the large response near resonance.

If $\cos \omega_0 t$ is a natural mode, use $y_p = t[A \cos \omega_0 t + B \sin \omega_0 t]$.



Worked example — undamped resonance trial

For $y'' + y = \cos t$, the homogeneous solution already contains $\cos t$ and $\sin t$, so $A\cos t + B\sin t$ would duplicate y_h . Multiply by t and choose the convenient trial $y_p = Ct\sin t$. Substitution gives $2C\cos t = \cos t$, hence $C = 1/2$:

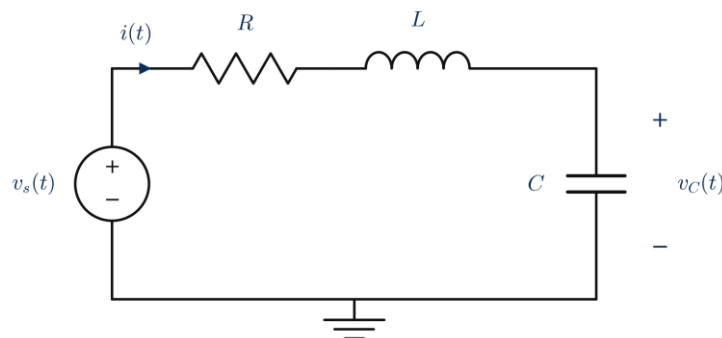
$$y_p = \frac{t}{2} \sin t$$

Result: the factor t makes the ideal resonant envelope grow linearly.

Engineering qualification: resistive loss limits resonance. The unbounded result belongs to the undamped mathematical model, not to a passive circuit with finite resistance.

10. Consolidation: roots, damping and response shape

Second-order analysis becomes much easier once you read the characteristic roots before solving for constants. Root location predicts the qualitative natural response immediately.



Series RLC circuit: the classic second-order system.

Roots	Natural response
two negative real	sum of two decaying exponentials
repeated negative real	$(A + Bt)e^{rt}$
$-\alpha \pm j\omega_d$	decaying sinusoid
$\pm j\omega_0$	undamped sinusoid

The root plane summarises the behaviour

Two negative real roots mean overdamped decay; a repeated negative root means critical damping; a left-half-plane conjugate pair means damped oscillation; purely imaginary roots mean ideal undamped oscillation; positive real parts mean growth. Then add the forcing and initial conditions. Initial conditions choose the coefficients, but they cannot turn real roots into oscillation or remove the envelope from a complex pair, so an overdamped root set must never produce persistent ringing.

Standard form $y'' + 2\alpha y' + \omega_0^2 y = 0$: compare α^2 with ω_0^2 to classify the roots.

Accompany each classification with a sketch: mark the decay rate, the oscillation rate and the expected envelope, then compare it with your closed-form solution. Sign errors in the roots become visually obvious.

Worked example — degree-level checkpoint

For $y'' + 8y' + 25y = 5$ with $y(0) = 1$ and $y'(0) = 0$, classify the response and find the steady value.

$$r = \frac{-8 \pm \sqrt{64 - 100}}{2} = -4 \pm j3$$

The roots give an underdamped, stable transient, and

$$y_p = \frac{5}{25} = 0.2$$

Result: the complete response is a damped 3 rad/s sinusoid about the steady value 0.2.

Teach-back questions

1. Why does a second-order ODE normally need two initial conditions?
2. What does each characteristic root contribute?
3. Why does a repeated root need a $t e^{rt}$ term?
4. How do the real and imaginary parts of a root map to damping and oscillation?
5. Why is the complete response the sum of natural and forced parts?

Next chapter: The mathematical foundation is now complete. Ch. 6 begins deriving first-order ODEs directly from KCL, KVL and the R, L, C element laws.