

First-Order Ordinary Differential Equations

Ordinary Differential Equations for Electrical Engineers — Foundation Ch. 4 of 15

1. What a first-order ODE represents

A first-order ODE contains a first derivative but no higher derivative. One independent initial condition is normally needed to select a unique solution. In EE, first-order dynamics arise when a circuit has one independent energy-storage state.

$$\frac{dy}{dt} = F(t, y)$$

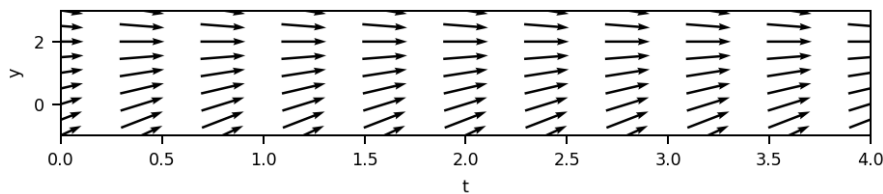
Solution family

Solving an ODE means finding a function $y(t)$, not a single number. Before an initial condition is applied, a first-order ODE usually has a one-parameter family of solutions.

One state variable, one starting value

A first-order ODE fixes the instantaneous slope once t and y are known: every point in the (t, y) plane has a preferred tangent direction. The equation does not give $y(t)$ directly; it gives the slope every valid solution must have. Choosing an initial point selects one trajectory, which then follows the local slopes. In circuits, one independent energy-storage state, such as a capacitor voltage or an inductor current, leads naturally to first-order dynamics.

General form: $y' = F(t, y)$, with initial condition $y(t_0) = y_0$.



Worked example — verify a proposed solution

Verify that $y = Ke^{-3t}$ satisfies $y' + 3y = 0$.

$$y' = -3Ke^{-3t}, \quad y' + 3y = -3Ke^{-3t} + 3Ke^{-3t} = 0$$

Result: every value of K gives a valid solution; an initial condition determines K .

Verification habit: differentiate the proposed solution, factor common terms and substitute back into the ODE. A zero residual for every t confirms membership of the solution family and quickly exposes sign errors.

2. Separable equations and direct integration

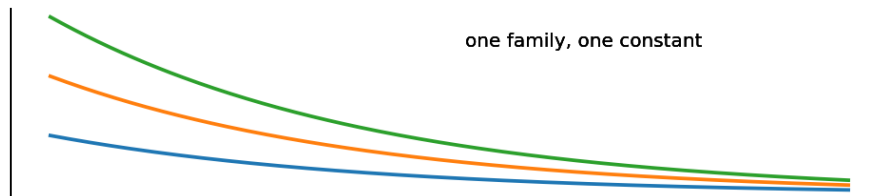
An ODE is separable when every factor involving y can be moved to one side and every factor involving t to the other. Each side is then integrated.

$$\frac{dy}{dt} = g(t)h(y) \quad \Rightarrow \quad \frac{1}{h(y)} dy = g(t) dt$$

Why separation works

Each side becomes a one-variable integral. The method is valid only where the divisions make sense: dividing by $h(y)$ can discard equilibrium solutions with $h(y) = 0$, so check those separately. After integrating, simplify, combine constants, and only then apply the initial condition. Exponentials appear frequently because $\int dy/y = \ln|y|$.

$$\int \frac{dy}{h(y)} = \int g(t) dt + C$$



Worked example — exponential decay by separation

Solve $y' = -4y$ with $y(0) = 10$.

$$\frac{1}{y} dy = -4 dt \Rightarrow \ln|y| = -4t + K \Rightarrow y = Ae^{-4t}$$

Applying $y(0) = 10$ gives $A = 10$:

$$y(t) = 10e^{-4t}$$

Result: the time constant is $\tau = 1/4$ s, and the solution decays monotonically from 10 towards zero.

Where the logarithm came from: separation creates dy/y , which integrates to $\ln|y|$. Exponentiating turns the additive constant into a multiplicative amplitude. This is the mathematical origin of the ubiquitous first-order natural response.

3. Standard first-order linear form

A first-order linear ODE can be written in the standard form below. The dependent variable and its derivative appear only to the first power and are never multiplied together.

$$y' + p(t)y = q(t)$$

Equation	Linear?	Reason
$y' + 3y = \sin t$	yes	y and y' to the first power only
$y' + ty = e^{-t}$	yes	a coefficient may depend on t
$y' + y^2 = 1$	no	y^2 term
$yy' + y = 0$	no	product $y \cdot y'$

Separate the coefficient from the forcing

$p(t)$ describes how the present state feeds back into its own rate of change; $q(t)$ is the prescribed forcing. The coefficient of y' must be exactly one before the integrating-factor formula is applied. If $p = a$ is a positive constant, the natural response contains e^{-at} , so a is the decay rate and $1/a$ the time constant.

The signs already predict stability: $a > 0$ gives the decaying mode e^{-at} , while $a < 0$ gives growth. In a passive circuit the stable sign is expected, so this can expose a KCL/KVL sign error before any solving.

Worked example — put an equation into standard form

Rewrite $5y' + 10y = 20u(t)$ in standard form by dividing every term by 5:

$$y' + 2y = 4u(t)$$

Result: $p(t) = 2$ and $q(t) = 4u(t)$; for $t > 0$ the forcing is the constant 4.

Classification first: confirm the equation is linear, normalise the coefficient of y' , and identify $p(t)$ and $q(t)$ with their units before choosing a method.

4. Integrating factors from first principles

The integrating-factor method turns $y' + p(t)y = q(t)$ into the derivative of a product. The factor is chosen so that the product rule reproduces exactly the left-hand side.

$$\mu(t) = e^{\int p(t) dt}$$

$$\frac{d}{dt}[\mu y] = \mu q(t)$$

Why the integrating factor works

We want $\mu y' + \mu p y$ to equal $d(\mu y)/dt = \mu' y + \mu y'$. Comparing terms gives $\mu' = p\mu$, hence $\mu = e^{\int p dt}$. Multiplying the whole ODE by μ makes the left side integrable in one step. The method is systematic and works even when $p(t)$ is not constant.

You can check any integrating factor without memorising the formula: it must satisfy $\mu' = p\mu$, and expanding $(\mu y)'$ must reproduce the left side of the multiplied ODE exactly. If not, the factor or the normalisation is wrong.



Worked example — solve $y' + 2y = 6$

Find $y(t)$ for $y' + 2y = 6$ with $y(0) = 1$.

$$\mu = e^{\int 2 dt} = e^{2t}, \quad \frac{d}{dt}(e^{2t}y) = 6e^{2t}$$

$$e^{2t}y = 3e^{2t} + K \Rightarrow y = 3 + Ke^{-2t}$$

Applying $y(0) = 1$ gives $K = -2$:

$$y(t) = 3 - 2e^{-2t}$$

Result: the steady value is 3, and the transient $-2e^{-2t}$ decays with $\tau = 0.5$ s.

Check: substituting into $y' + 2y = 6$ confirms both the constant particular term and the decaying natural term.

5. Natural and forced solution components

For a linear non-homogeneous ODE, the complete solution is the sum of a homogeneous (natural) solution and a particular (forced) solution. This decomposition is central to EE transient analysis.

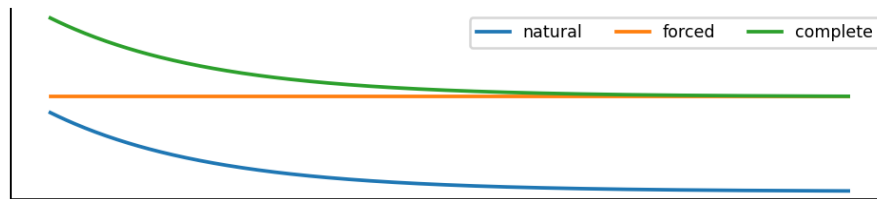
$$y(t) = y_h(t) + y_p(t)$$

A memory part and a driven part

The homogeneous solution y_h describes how the stored initial state evolves with the forcing set to zero. The particular solution y_p is one response produced by the input. Their sum satisfies the full equation, and the initial condition is applied to the sum, not to y_p alone. In a stable first-order system the natural term decays, leaving only the forced behaviour.

The split is not unique until a particular solution is chosen, but the total response is. Engineers usually pick y_p as the steady-state response, so the natural term represents the transient error decaying from the initial condition.

With the linear operator $D[y] = y' + p(t)y$: if $D[y_h] = 0$ and $D[y_p] = q(t)$, then $y = y_h + y_p$ satisfies $D[y] = q(t)$.



Worked example — separate the complete response

Solve $y' + 5y = 20$ with $y(0) = 1$. The homogeneous equation $y' + 5y = 0$ gives $y_h = Ke^{-5t}$. For constant forcing try $y_p = A$: then $5A = 20$, so $A = 4$.

$$y = 4 + Ke^{-5t}, \quad y(0) = 1 \Rightarrow K = -3$$

$$y(t) = 4 - 3e^{-5t}$$

Result: the forced (steady) part is 4; the natural (transient) part is $-3e^{-5t}$.

Important: "natural" describes the homogeneous dynamics, not a source-free experiment. Even with a source connected, the complete response needs a natural component to satisfy the initial condition.

6. Initial-value problems and uniqueness

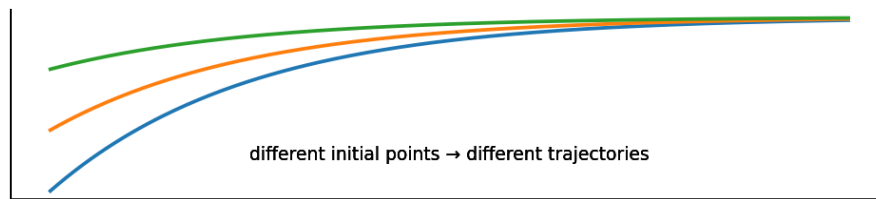
An initial-value problem (IVP) specifies both the differential equation and a starting value. For well-behaved first-order linear ODEs this determines one unique solution.

$$y' + p(t)y = q(t), \quad y(t_0) = y_0$$

Why trajectories do not cross

If $F(t, y)$ is well behaved near the initial point, $y' = F(t, y)$ with $y(t_0) = y_0$ has exactly one local solution; for the linear IVP, continuous p and q on an interval guarantee a unique solution there. Two solution curves cannot cross at a regular point, because the equation assigns only one slope there. Uniqueness is what makes a

deterministic lumped model useful: present state plus input determine the future. If a sketch of candidate solutions crosses, at least one curve is wrong.



Worked example — same ODE, different initial state

Compare $y' + 2y = 4$ for $y(0) = 0$ and $y(0) = 5$. The general solution is $y(t) = 2 + Ke^{-2t}$.

Case A: $K = -2$, $y_A = 2 - 2e^{-2t}$

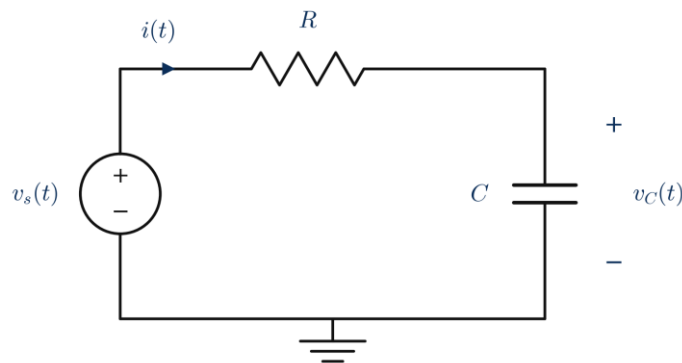
Case B: $K = 3$, $y_B = 2 + 3e^{-2t}$

Result: both approach the forced value 2, but their transients differ because the initial state differs. Their difference satisfies the homogeneous equation and decays, so the curves approach each other without ever crossing.

Physical interpretation: the ODE encodes the system law; the initial condition encodes where the system starts. A source alone does not determine the transient.

7. Constant coefficients and the universal exponential response

Most first-order lumped circuits with fixed components reduce to a constant-coefficient ODE. The solution can then be written directly in "initial-to-final" form once the time constant and final value are known.



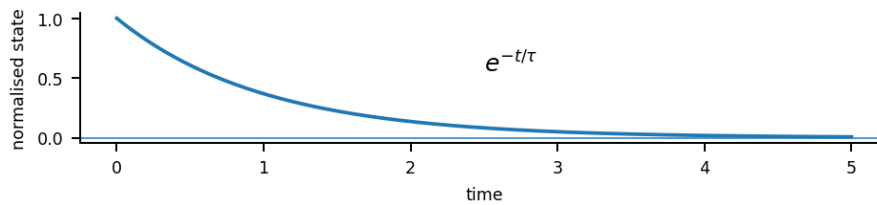
Series RC circuit: a typical first-order, constant-coefficient system.

$$y(t) = y_{\infty} + (y_0 - y_{\infty})e^{-t/\tau}$$

Reading the response without re-solving

For $\tau y' + y = Kx_0$ with a constant input, the final value is $y_{\infty} = Kx_0$, and the deviation from it obeys a source-free equation. The response is therefore "final value plus initial error times a decaying exponential". Two checks are built in: at $t = t_0$ the exponential is 1, giving the initial value; at long times it vanishes, giving the final value.

$$y(t) = y_f + [y(t_0) - y_f]e^{-(t-t_0)/\tau}$$



Worked example — RC charging form

An RC circuit has $\tau = 10$ ms, $v_C(0^+) = 2$ V and final value 8 V. Write $v_C(t)$ and find $v_C(10$ ms).

$$v_C(t) = 8 + (2 - 8) e^{-t/0.01} = 8 - 6e^{-100t}$$

$$v_C(0.01) = 8 - 6e^{-1} = 5.79 \text{ V}$$

Result: after one time constant the capacitor has completed 63.2% of the 6 V change (36.8% of the error remains).

Shortcut with meaning: this formula does not replace deriving the ODE. It is valid for a stable first-order constant-coefficient system under constant forcing.

8. Piecewise forcing and switching times

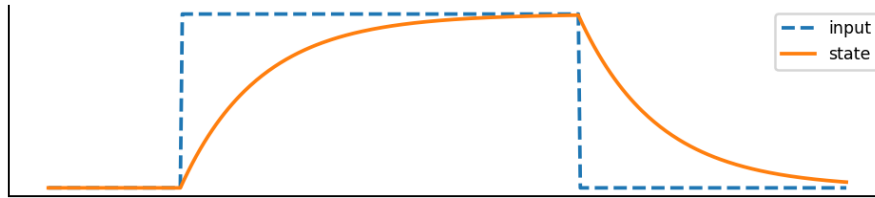
Real circuits switch. When the input changes at $t = t_s$, solve the ODE separately on each interval and use the state at the switching instant as the initial condition for the next.

$$x(t) = A[u(t) - u(t - T)]$$

Solve interval by interval

Write one solution per interval; the end state of one interval is the initial state of the next. In ordinary finite switching, capacitor voltage and inductor current are continuous, while other variables may jump. The state y may be continuous even though the forcing, and therefore y' , changes instantly. Draw the input timeline first and mark every switching instant. Resetting the local time origin can simplify the algebra, but keep the physical switching time clear.

$$\text{For } t \geq t_s: y(t) = y_f^{(2)} + [y(t_s^+) - y_f^{(2)}] e^{-(t-t_s)/\tau_2}$$



Worked example — first-order pulse response

A system obeys $y' + y = 2x(t)$ with $y(0) = 0$, where $x = 1$ for $0 \leq t < 1$ s and $x = 0$ afterwards. For $0 \leq t < 1$ the final value is 2:

$$y(t) = 2(1 - e^{-t})$$

At $t = 1^-$, $y = 2(1 - e^{-1}) = 1.264$. For $t \geq 1$ the forcing is zero, and this value is the new initial condition:

$$y(t) = 1.264 e^{-(t-1)}, \quad t \geq 1$$

Result: the state is continuous at $t = 1$ even though the input switches abruptly.

Switching habit: use $(t - t_s)$ in the exponential. Using $e^{-t/\tau}$ blindly after a later switch, or restarting from the original initial condition, gives the wrong answer.

9. Step, exponential and sinusoidal forcing

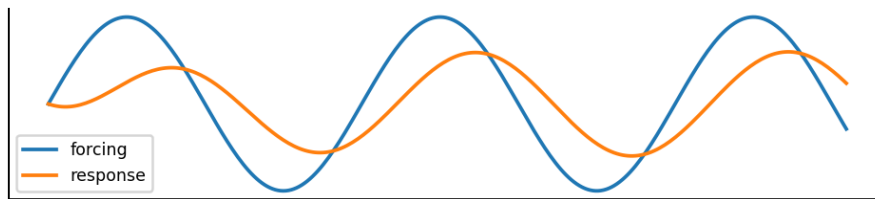
A linear first-order ODE can be driven by many waveforms. The particular-solution trial should resemble the forcing, unless that trial duplicates a natural mode.

Forcing $q(t)$	Typical trial y_p
constant A	constant K
Ae^{at}	Ke^{at}
$A\cos\omega t + B\sin\omega t$	$C\cos\omega t + D\sin\omega t$

Choosing the particular trial

For e^{at} , try Ke^{at} unless a is already a natural root. For sinusoidal forcing try $C\cos\omega t + D\sin\omega t$, or use a complex exponential and take the real part at the end. The forcing changes only the particular part; the natural response still comes from the homogeneous equation, and the initial condition is applied to $y_h + y_p$.

A sinusoidal trial needs both sine and cosine even if the forcing has only one, because differentiation turns one into the other. Solving for both coefficients is equivalent to finding the amplitude and phase shift of the steady-state response.



Worked example — sinusoidal forcing

Find a particular solution of $y' + y = \cos t$. Try $y_p = A \cos t + B \sin t$, so $y_p' = -A \sin t + B \cos t$. Substituting and collecting terms:

$$(A + B) \cos t + (B - A) \sin t = \cos t$$

Hence $A + B = 1$ and $B - A = 0$, giving $A = B = 1/2$:

$$y_p = \frac{1}{2} \cos t + \frac{1}{2} \sin t$$

Result: the complete response adds Ke^{-t} . The forced part has the input frequency but a different amplitude and phase; the transient decays and the waveform settles into a steady sinusoid.

10. Consolidation and degree-level checkpoint

A first-order workflow should now be systematic: classify the equation, choose a method, apply the initial condition, then interpret the transient and forced parts.

Method selection

Situation	Useful method
variables separate cleanly	separation of variables
general linear $y' + p(t)y = q(t)$	integrating factor
constant coefficient, simple forcing	natural + particular
stable first-order DC switch	initial-to-final exponential form

Predict before the algebra finishes

Classify the equation, find the time constant, identify the forcing and the final value, then sketch the expected response. A stable first-order response cannot ring, because it has only one real natural mode. Also compute the initial slope directly from the ODE by substituting t_0 and y_0 , and compare it with the derivative of your closed-form answer; this catches wrong constants that still satisfy the initial value.

Three checks: $y(t_0) = y_0$; zero ODE residual; stable limit equals the forcing equilibrium.

Worked example — degree-level checkpoint

Solve $0.02 y' + y = 10$ with $y(0) = 4$; identify τ , the initial and final values, and the transient.

$$y' + 50y = 500 \quad \Rightarrow \quad y(t) = 10 + Ke^{-50t}$$

$$4 = 10 + K \quad \Rightarrow \quad K = -6, \quad y(t) = 10 - 6e^{-50t}$$

Result: $\tau = 0.02 \text{ s}$, $y(0) = 4$, $y(\infty) = 10$, transient $-6e^{-50t}$.

Method statement: classify, identify $p(t)$ and $q(t)$, predict the natural response, choose a particular form, apply the initial condition, verify. This sequence applies to almost every first-order problem in the circuit chapters.

Teach-back questions

1. Why does a first-order ODE normally need one initial condition?
2. What makes an equation separable?
3. Why does the integrating factor work?
4. What distinguishes the natural response from the forced response?
5. How should a solution be restarted after a switching event?

Next chapter: Second-order ODEs introduce two natural modes, damping classifications, oscillation and resonance.