

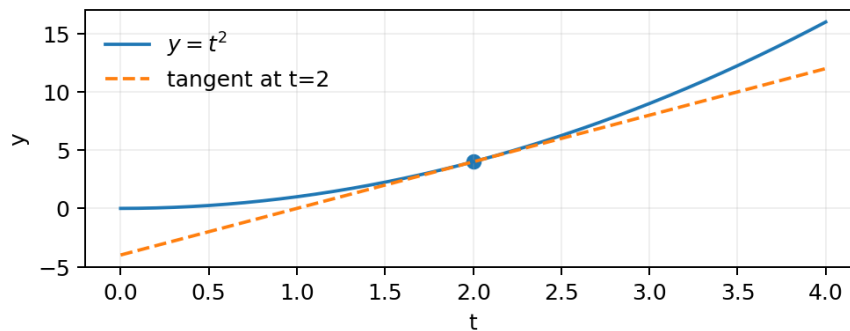
Differentiation and Integration Techniques

Ordinary Differential Equations for Electrical Engineers — Foundation Ch. 3 of 15

1. Limits, continuity and the derivative as local slope

A derivative measures how rapidly one quantity changes with another. In circuit ODEs the independent variable is usually time, so derivatives describe rates such as voltage change, current change or charge flow. The limit definition explains what "instantaneous" means:

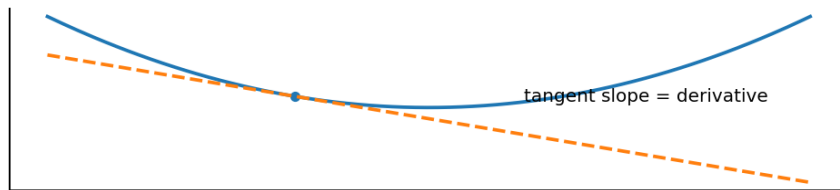
$$\frac{dy}{dt} = \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t}$$



A derivative is a limiting slope

The average slope over a finite interval is $\Delta y / \Delta t$; the derivative is the value this slope approaches as the interval shrinks around a point. Differentiation is therefore local: $y'(t_0)$ depends only on behaviour arbitrarily close to t_0 , not on the area under the curve. Continuity alone does not guarantee differentiability: a continuous waveform can have a sharp corner where the left and right slopes disagree, and switching models often contain exactly such points.

At a switching instant, ask separately whether the waveform is continuous and whether its derivative is continuous. A capacitor voltage may stay continuous while its slope changes abruptly because the current changes.



Worked example — instantaneous slope

For $y = t^2$, find dy/dt and the slope at $t = 2$ s.

$$\frac{dy}{dt} = 2t, \quad \left. \frac{dy}{dt} \right|_{t=2} = 4$$

Result: the tangent slope at $t = 2$ is 4 units of y per second.

Interpretation: the tangent line is a local model of the curve. It matches the function at the chosen point and gives the best first-order prediction for a small change in t ; the approximation worsens further from the point.

Continuity: a differentiable function is continuous at that point. In ideal switching circuits some source waveforms can jump, but capacitor voltage and inductor current obey additional physical continuity constraints.

2. Core derivative rules

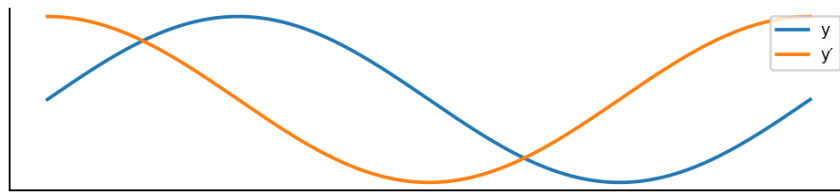
A small set of derivative rules covers most functions in introductory ODEs. Memorise the patterns, but always check units and signs.

Function	Derivative
t^n	$n t^{n-1}$
e^{at}	$a e^{at}$
$\ln t$	$1/t$
$\sin \omega t$	$\omega \cos \omega t$
$\cos \omega t$	$-\omega \sin \omega t$

Learn the patterns, then check dimensions

Differentiating e^{at} multiplies by a ; differentiating a sinusoid multiplies by the angular frequency and shifts the phase. If t is in seconds, a and ω must have units s^{-1} so that at and ωt are dimensionless.

A quick verification: differentiating a sinusoid should multiply its amplitude by ω and advance its phase by 90° ; differentiating an exponential should preserve its shape. If your result breaks either pattern, re-check the chain-rule factor.



Worked example — differentiate an EE waveform

Differentiate $x(t) = 3e^{-5t} + 2\cos 100t$.

$$\frac{dx}{dt} = -15e^{-5t} - 200\sin 100t$$

Result: each term is differentiated independently; the chain-rule factor 100 appears in the sinusoidal term.

Check: polynomial powers fall by one, exponentials keep their shape and sinusoids change phase. Every term must have the same units.

The derivative as an operator

d/dt is an instruction. Acting on a sum, it passes through each term; this linearity is one reason linear ODEs are so manageable.

$$\frac{d}{dt}[af(t) + bg(t)] = af'(t) + bg'(t)$$

3. Product, quotient and chain rules

Many trial solutions combine functions. The product rule handles products, the quotient rule handles ratios, and the chain rule handles a function of another function. The chain rule is especially important for exponentials and sinusoids whose arguments contain a time scale.

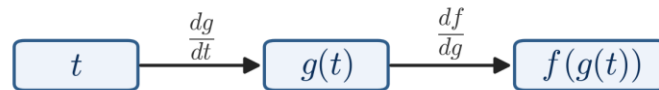
$$\frac{d}{dt}[fg] = f'g + fg'$$

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dt}F(g(t)) = \frac{dF}{dg} \frac{dg}{dt}$$

Track where the time dependence sits

Many EE waveforms need more than one rule at once, for example $t^2 e^{-t/\tau}$ or $\sin(\omega t + \phi(t))$. Identify the outer operation first, then work inward, and write the unchanged factor explicitly at each step: most mistakes are omissions, not difficult calculus. For products, make sure both the "differentiate-left" and "differentiate-right" terms appear.



Worked example — polynomial times exponential

Differentiate $y(t) = t e^{-2t}$.

$$y' = 1 \cdot e^{-2t} + t(-2e^{-2t}) = e^{-2t}(1 - 2t)$$

Result: the product rule gives two terms; factorising the common exponential simplifies later ODE substitution.

Common trap: $\frac{d}{dt} e^{-2t} = -2e^{-2t}$, not e^{-2t} . The inner derivative is essential; missing either product term changes both the initial slope and the long-time behaviour.

4. Higher derivatives and physical interpretation

A second derivative is the derivative of the first derivative. In mechanics it is acceleration; in circuits it usually appears because two independent energy-storage mechanisms create second-order dynamics.

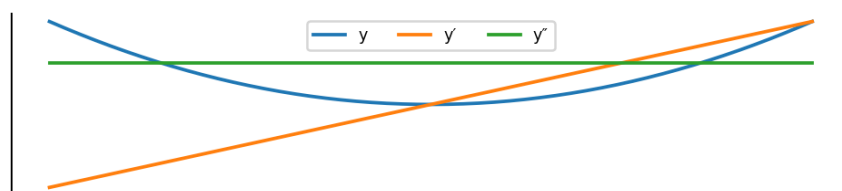
$$y' = \frac{dy}{dt}, \quad y'' = \frac{d^2y}{dt^2}$$

Successive layers of dynamics

y' is a rate and y'' is a rate of a rate; the units always tell you what each means. A second-order ODE carries enough dynamic information to need two independent initial quantities, which fix the two constants in the solution. The sign of y'' gives curvature: positive bends the graph upward, negative bends it downward.

For $y = Ae^{st}$: $y' = sAe^{st}$ and $y'' = s^2Ae^{st}$. For a pure sinusoid, $y'' = -\omega^2 y$.

These identities let you verify second-order trial solutions almost by inspection.



Worked example — first and second rates

Let $y(t) = 5e^{-3t}$. Find y' and y'' and compare their units.

$$y' = -15e^{-3t}, \quad y'' = 45e^{-3t}$$

Result: if y is in volts, y' is in V/s and y'' in V/s². y' says whether the quantity is rising or falling; y'' says how that rate is changing.

Why second derivatives arise

Combining two storage-element relations can introduce a second derivative. A series RLC equation written for charge q contains q'' , because current is q' and the inductor voltage is proportional to $di/dt = q''$:

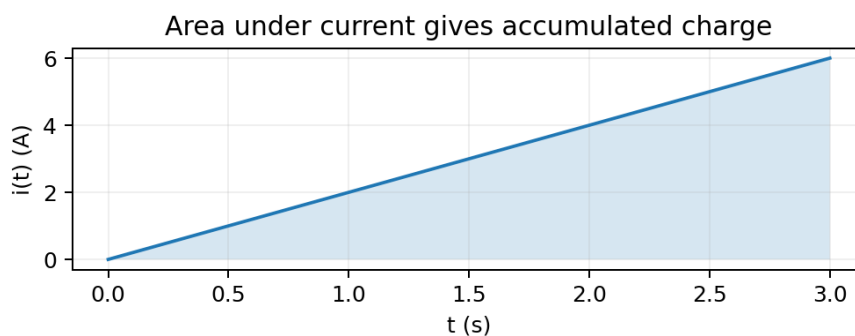
$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = v_s(t)$$

Interpretation: Order counts independent state information. A second-order model normally needs two independent initial conditions.

5. Antiderivatives and definite integrals

Integration reverses differentiation and also represents accumulation. Current is the rate of flow of charge, so integrating current over time gives the charge transferred.

$$\int f'(t) dt = f(t) + K$$



A definite integral is an accumulated rate

The area under an i - t graph has units $A \cdot s = C$. The lower limit anchors the accumulation at a known time and removes the unexplained constant. Signed area matters: current below the time axis removes charge. A current pulse that changes sign can move substantial charge each way yet give a small net integral, so sketch piecewise waveforms before integrating.

$$q(t) = q(t_0) + \int_{t_0}^t i(\lambda) d\lambda$$

Worked example — charge from current

A current $i(t) = 2t$ A flows from $t = 0$ to 3 s. Find the transferred charge.

$$q = \int_0^3 2t dt = [t^2]_0^3 = 9 \text{ C}$$

Result: the area under the current–time curve is 9 coulombs.

Units check: $A \cdot s = C$; the differential dt changes the units during accumulation. A constant current would make the charge change linearly; a decaying current makes it approach a finite asymptote.

6. Integrating exponential and sinusoidal functions

The antiderivatives used most in ODE work mirror the derivative rules. Keep the constant for indefinite integrals, because differentiation removes constants.

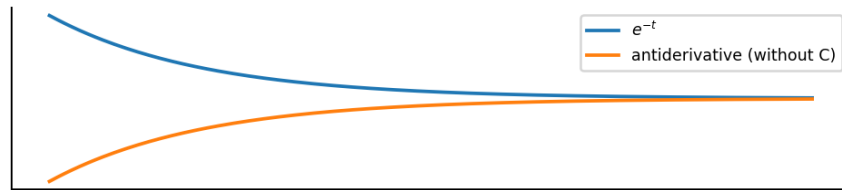
$$\int e^{at} dt = \frac{1}{a} e^{at} + K \quad (a \neq 0)$$

$$\int \cos \omega t dt = \frac{1}{\omega} \sin \omega t + K$$

$$\int \sin \omega t dt = -\frac{1}{\omega} \cos \omega t + K$$

Integrate by reversing the derivative pattern

Since $d(e^{at})/dt = ae^{at}$, integration divides by a ; since $d(\sin \omega t)/dt = \omega \cos \omega t$, integrating $\cos \omega t$ divides by ω . Include the constant for an indefinite integral and fix it from the initial condition, or use definite limits to write the initial value directly. For a shifted sinusoid $\cos(\omega t + \phi)$ the phase is unchanged by integration. Never drop the factor $1/\omega$: the integral must gain one factor of time in its units.



Worked example — capacitor voltage from exponential current

A $10 \mu\text{F}$ capacitor carries $i(t) = 2e^{-100t}$ mA with $v_C(0) = 1$ V. Find $v_C(t)$.

$$v_C(t) = v_C(0) + \frac{1}{C} \int_0^t i(\tau) d\tau = 1 + \frac{2 \times 10^{-3}}{10 \times 10^{-6}} \int_0^t e^{-100\tau} d\tau$$

$$v_C(t) = 1 + 2(1 - e^{-100t}) = 3 - 2e^{-100t} \text{ V}$$

Result: v_C rises from 1 V towards 3 V.

Check: differentiate the result and multiply by C ; you must recover the original current exactly. This tests the whole function, not just one value.

7. Substitution and integration by parts

Substitution simplifies a composite integrand by changing variables. Integration by parts is the integral counterpart of the product rule and suits products such as $t e^{-at}$.

$$\int f(g(t)) g'(t) dt = \int f(u) du$$

$$\int u dv = uv - \int v du$$

Choosing the method

Substitution reverses the chain rule: use it when an inner function and its derivative appear together. Integration by parts reverses the product rule: use it when one factor simplifies on differentiation and the other is easy to integrate. Choose u to simplify. If the remaining integral is as hard as the original, the choice of u and dv was probably poor.

Worked example — integrate $t e^{-2t}$

Choose $u = t$ and $dv = e^{-2t} dt$, so $du = dt$ and $v = -1/2 e^{-2t}$.

$$\int t e^{-2t} dt = -\frac{t}{2} e^{-2t} + \frac{1}{2} \int e^{-2t} dt = -\frac{t}{2} e^{-2t} - \frac{1}{4} e^{-2t} + K$$

Result: a polynomial times an exponential stays an exponential times a polynomial after integration.

Check: differentiate the answer; the product-rule terms collapse back to $t e^{-2t}$.

Why it matters: terms like $t e^{rt}$ arise for repeated roots and resonant forcing in later chapters.

8. Initial conditions and definite-integral forms

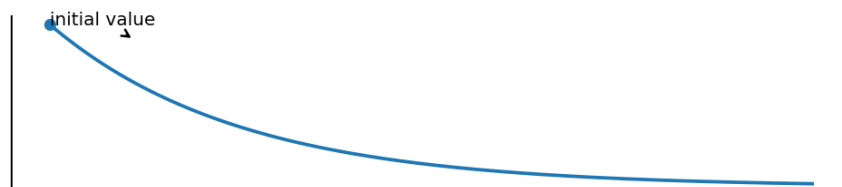
An indefinite integral gives a family of functions because of the constant K . An initial condition selects the member that matches the circuit's physical state at the reference time.

$$\frac{dy}{dt} = f(t) \quad \Rightarrow \quad y(t) = y(t_0) + \int_{t_0}^t f(\tau) d\tau$$

Initial conditions are part of the calculus

The constant restores the starting level that differentiation erased. The definite-integral form is safer because the condition is built in from the start. At a switching time t_s , use the state just after the switch as the lower-limit value and integrate the post-switch rate law from there; never carry an old integration constant across a switching instant.

$$y(t) = y(t_s^+) + \int_{t_s}^t y'(\lambda) d\lambda$$

**Worked example — recovering the integration constant**

Given $dy/dt = 6t$ and $y(1) = 5$, find $y(t)$.

$$y(t) = 3t^2 + K, \quad 5 = 3(1)^2 + K \Rightarrow K = 2$$

$$y(t) = 3t^2 + 2$$

Result: the initial value removes the arbitrary constant and makes the solution unique. Apply the condition only after writing the general antiderivative, then verify the function at that time.

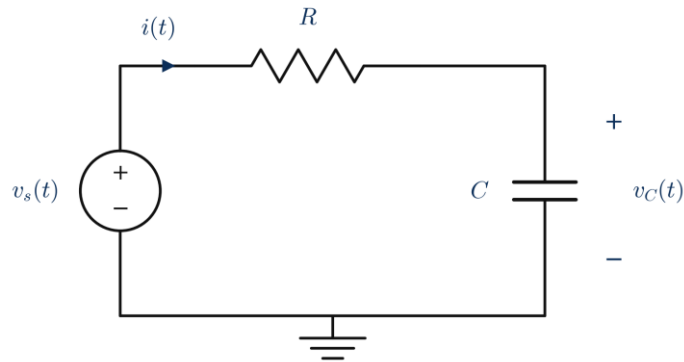
Dummy integration variables

Inside a definite integral use τ rather than t as the integration variable, so that the upper limit t stays visibly distinct:

$$y(t) = y(0) + \int_0^t f(\tau) d\tau$$

9. Calculus with charge, current, voltage and energy

The capacitor and inductor laws are not arbitrary: they express how stored electric and magnetic energy change with the circuit variables. They are the bridge from calculus to circuit ODEs.



Series RC circuit: the capacitor voltage is the stored state.

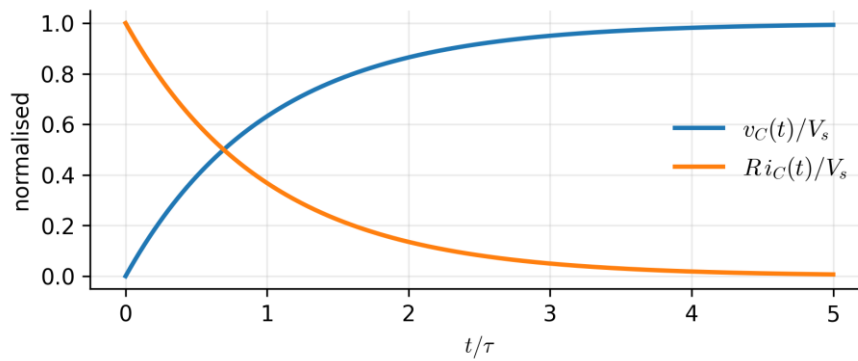
$$i_C = C \frac{dv_C}{dt}, \quad v_L = L \frac{di_L}{dt}$$

$$w_C = \frac{1}{2} C v_C^2, \quad w_L = \frac{1}{2} L i_L^2$$

Element laws are calculus statements

Each law links a stored state to its time derivative: capacitor current is proportional to the rate of change of voltage, and inductor voltage to the rate of change of current. Integrating the laws gives the change in stored state, and the energy formulas provide an independent physical check.

The laws also impose continuity. A finite capacitor current gives a finite dv_C/dt , so v_C cannot jump; a finite inductor voltage gives a finite di_L/dt , so i_L cannot jump. These become powerful initial-condition tools in later chapters.



Worked example — current from a capacitor voltage

For $C = 4 \mu\text{F}$ and $v_C(t) = 5(1 - e^{-200t}) \text{ V}$, find $i_C(t)$.

$$\frac{dv_C}{dt} = 1000 e^{-200t} \text{ V/s}$$

$$i_C = C \frac{dv_C}{dt} = 4 \times 10^{-6} (1000 e^{-200t}) = 4 e^{-200t} \text{ mA}$$

Result: the charging current starts at 4 mA and decays to zero as the capacitor reaches steady state.

Sign convention: $i_C = C dv_C/dt$ holds when the current enters the positive-voltage terminal. Reversing either reference changes the sign, so draw the polarity and current arrow before differentiating.

10. Consolidation and degree-level checkpoint

Calculus becomes useful in EE when every derivative and integral is tied to a physical quantity, a unit and an initial value. The goal is interpretation, not symbolic manipulation alone.

Core patterns

Pattern	Meaning
dy/dt	instantaneous rate of change
d^2y/dt^2	rate of change of the rate
$\int f dt$	accumulation (antiderivative)
initial value	selects a unique member of the solution family
chain rule	accounts for scaled or composite arguments

Three ways to catch calculus errors

First, differentiate your antiderivative and confirm that the integrand returns. Second, check units: a derivative divides by time and an integral multiplies by time. Third, use initial or limiting values. In circuits also remember continuity: ideal capacitor voltage and inductor current cannot jump without an impulse-like excitation.

Verification pattern: derivative check + units check + initial/final-value check.

One line differentiating your result, one applying the initial condition and one checking units can earn method credit even if later arithmetic slips, and give three independent chances to catch an error.

Worked example — combined calculus check

A 2 H inductor carries $i(t) = 3 - 3e^{-4t}$ A. Find $v_L(t)$, $i(0)$ and $i(\infty)$.

$$\frac{di}{dt} = 12 e^{-4t} \text{ A/s}, \quad v_L = L \frac{di}{dt} = 24 e^{-4t} \text{ V}$$

$$i(0) = 0, \quad i(\infty) = 3 \text{ A}$$

Result: the inductor voltage is largest just after switching and vanishes once the current is constant. State what each derivative or integral means physically, with units and initial or final values; that habit makes circuit ODE solutions much easier to debug.

Teach-back questions

1. Why is a derivative a local quantity but a definite integral an accumulated one?
2. Why does differentiating e^{at} reproduce the same function?
3. Why do second-order models need two initial conditions?
4. How do $i_C = C dv_C/dt$ and $v_L = L di_L/dt$ encode energy storage?
5. Why is τ often used as a dummy integration variable?

Next chapter: We now use these tools to solve first-order ODEs directly in the time domain.