

Algebra, Exponentials, Logarithms and Complex Numbers

Ordinary Differential Equations for Electrical Engineers — Foundation Ch. 2 of 15

1. Algebraic rearrangement and rational expressions

ODE work is full of algebra. Long before the calculus becomes difficult, mistakes arise from signs, fractions and rearrangement. The aim is to manipulate equations without changing their meaning, and to keep engineering units attached to the quantities.

Rearranging without losing structure

Treat derivative terms as complete algebraic objects. If a coefficient multiplies the whole derivative, divide every term consistently. Factor before cancelling, and cancel factors rather than individual terms.

$$RC \frac{dv}{dt} + v = v_s(t) \quad \Rightarrow \quad \frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} v_s(t)$$

Operation	Safe habit
collect terms	move a complete term, including its sign
divide	divide every term on both sides
cancel	cancel common factors, not added terms
fractions	use a common denominator before adding

Algebra is part of the modelling

Rearrangement should preserve the physical grouping of terms. If RC multiplies dv/dt , then RC is one coefficient with units of seconds; dividing the whole equation by RC creates the rate constant $1/(RC)$. Never cancel across addition, and never move a term without its sign. After every division, ask whether every term still has the same units. Keeping a group such as RC intact until the last step is often more informative, because it immediately identifies a time constant.

Normalised first-order form: $y' + ay = g(t)$, where a has units s^{-1} when t is in seconds.

Safe algebra habit: Write one legal operation per line. If you divide by a parameter, show it acting on every term. Sign errors become visible and units are preserved.

Worked example — normalising a circuit-form ODE

Let $R = 2 \text{ k}\Omega$ and $C = 5 \text{ }\mu\text{F}$. Rewrite $RC \, dv/dt + v = v_s(t)$ in standard form and identify the coefficient of v . First, $RC = (2000)(5 \times 10^{-6}) = 0.010 \text{ s}$, so

$$0.01 \frac{dv}{dt} + v = v_s(t)$$

Divide every term by 0.01 s:

$$\frac{dv}{dt} + 100 v = 100 v_s(t)$$

Result: the coefficient 100 s^{-1} is the reciprocal time constant $1/(RC)$.

Check: the coefficient of v must have units s^{-1} , because it is added to dv/dt (units V/s).

Interpretation: after normalising, the coefficient of v is the decay rate $1/\tau$. Doubling R or C would halve this rate and slow the transient, so normalising reveals physical behaviour before the ODE is solved.

2. Powers, roots and engineering scaling

Powers and roots appear when solving characteristic equations, defining natural frequencies and converting between engineering prefixes. The exponent laws matter most when expressions contain powers of ten.

$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn}$$

$$\sqrt{x} = x^{1/2}, \quad \frac{1}{\sqrt{x}} = x^{-1/2}$$

Engineering prefixes

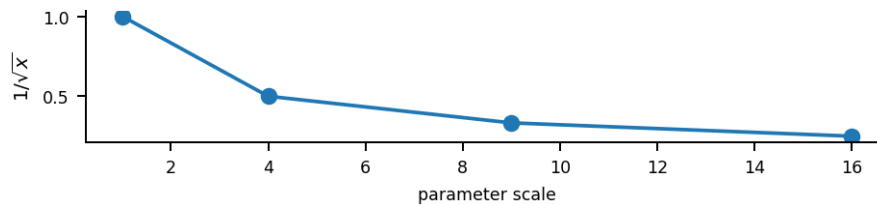
Prefix	Factor	Example
milli, m	10^{-3}	$1 \text{ ms} = 10^{-3} \text{ s}$
micro, μ	10^{-6}	$1 \text{ }\mu\text{F} = 10^{-6} \text{ F}$
kilo, k	10^3	$1 \text{ k}\Omega = 10^3 \Omega$
mega, M	10^6	$1 \text{ MHz} = 10^6 \text{ Hz}$

Powers tell you how a quantity scales

Exponent laws simplify square roots, inverse powers and scientific notation without changing the physical dependence. If a natural frequency is proportional to $1/\sqrt{LC}$, quadrupling L halves the frequency: the square root weakens the scaling. Negative exponents are reciprocals, so s^{-1} means "per second"; fractional exponents are roots. Keep units inside the same operation: $\sqrt{H \cdot F}$ has units of seconds, so $1/\sqrt{LC}$ has units s^{-1} .

$$a^m a^n = a^{m+n}, \quad a^m / a^n = a^{m-n}, \quad (a^m)^n = a^{mn}, \quad a^{-n} = 1/a^n$$

Scaling intuition is often more useful than a calculator. If a quantity varies as x^2 , doubling x multiplies it by four; as x^{-1} , doubling x halves it; as $x^{-1/2}$, quadrupling x halves it. Read the exponent as a sensitivity rule before substituting numbers.



Worked example — natural-frequency scaling

For $L = 10 \text{ mH}$ and $C = 1 \text{ } \mu\text{F}$, find the undamped natural angular frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Here $LC = (10 \times 10^{-3})(1 \times 10^{-6}) = 10^{-8} \text{ s}^2$, so

$$\omega_0 = \frac{1}{\sqrt{10^{-8}}} = 10^4 \text{ rad/s}$$

Result: $\omega_0 = 10 \text{ krad/s}$, so $f_0 = \omega_0/2\pi \approx 1.59 \text{ kHz}$.

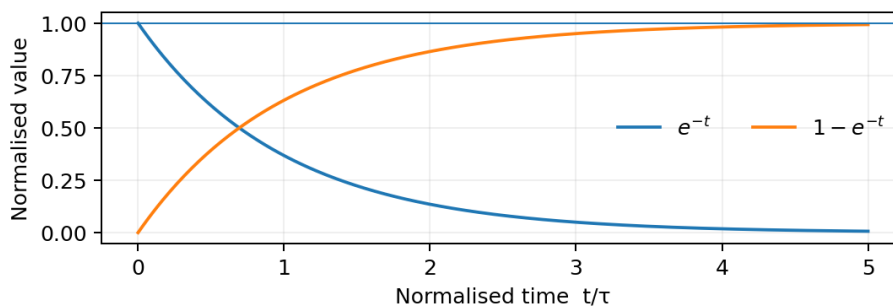
Common trap: square roots halve exponents: $\sqrt{10^{-8}} = 10^{-4}$, not 10^{-8} .

Interpretation: because of the square root, a large component change produces a smaller frequency change. For small variations, a percentage error in L or C shifts ω_0 by roughly half that percentage, in the opposite direction.

3. The natural exponential and growth or decay

The exponential is the natural language of systems whose rate of change is proportional to their present value. First-order passive circuits therefore produce exponentials whenever they release stored energy or approach a new equilibrium.

$$\frac{dy}{dt} = ay \quad \Rightarrow \quad y(t) = Ke^{at}$$



Time constant

For a stable first-order response, $a = -1/\tau$. At $t = \tau$ a decaying exponential has fallen to $e^{-1} \approx 0.368$ of its initial value, and a rising response $1 - e^{-t/\tau}$ has completed 0.632 of its final change.

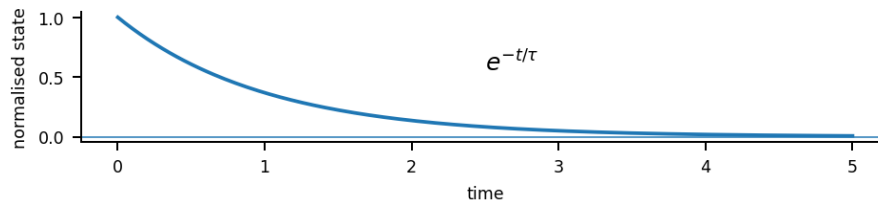
The exponent must be dimensionless

An exponential can only take a pure number as its argument. In time-domain ODEs, t/τ is dimensionless because both quantities are times, which makes this a powerful diagnostic. The sign sets the direction ($e^{-t/\tau}$

decays, $e^{+t/\tau}$ grows) and the size of $1/\tau$ sets the speed. "Fast" and "slow" transients are encoded directly in the coefficient of t .

For $y = Ae^{st}$, s must have units s^{-1} . Stable real decay requires $\text{Re}\{s\} < 0$.

A quick sketch helps: mark the initial value, one time constant and the asymptote. At $t = \tau$ about 63.2% of the total change is complete; by 5τ the response is within about 0.7% of its final value. These landmarks expose an implausible computed curve immediately.



Worked example — RC decay

A capacitor starts at 8 V and discharges through a resistor with $\tau = 20$ ms. Find the voltage after 40 ms.

$$v_C(t) = 8e^{-t/0.020}$$

$$v_C(0.040) = 8e^{-2} = 1.083 \text{ V}$$

Result: after two time constants 13.5% of the initial voltage remains, about 1.08 V.

Interpretation: the exponential never mathematically reaches zero, but after about 5τ less than 1% of the transient remains.

Check: $t = 0$ must reproduce the initial value, and a very large t must reproduce the expected final value.

4. Natural logarithms and solving exponential equations

The natural logarithm $\ln(\cdot)$ is the inverse of the exponential. It is the tool to use when the unknown sits in an exponent, for example when finding how long a transient takes to reach a given percentage.

$$y = e^x \Leftrightarrow x = \ln y$$

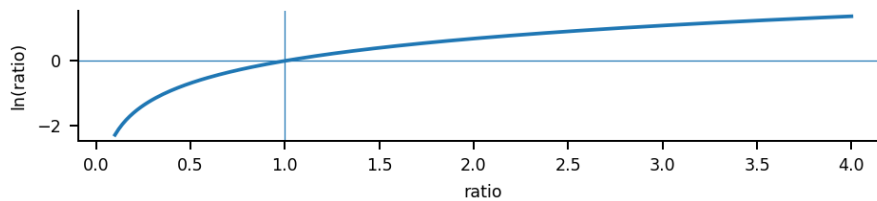
$$\ln(ab) = \ln a + \ln b, \quad \ln(a^n) = n \ln a$$

Logarithms undo exponentials

To solve for time, first isolate the exponential ratio, then take the logarithm of a dimensionless positive quantity. In decay problems the logarithm of a number between zero and one is negative, and the minus sign in the exponent then gives a positive time. A negative elapsed time usually means the ratio was inverted.

$$\text{If } y/y_0 = e^{-t/\tau}, \text{ then } \ln(y/y_0) = -t/\tau \text{ and } t = -\tau \ln(y/y_0).$$

For a decay from y_0 towards zero, $0 < y/y_0 < 1$ and $\ln(y/y_0) < 0$, so $-\tau \ln(y/y_0)$ is positive. If the ratio exceeds one in a passive decay calculation, revisit the model or the reference values.



Worked example — time to reach 10%

A decaying transient is $x(t) = x_0 e^{-t/\tau}$. When is only 10% of x_0 left?

$$0.1 x_0 = x_0 e^{-t/\tau}$$

Cancel x_0 and take natural logarithms:

$$\ln(0.1) = -\frac{t}{\tau} \Rightarrow t = -\tau \ln(0.1) = 2.303 \tau$$

Result: a first-order transient reaches 10% of its initial value at 2.303τ .

Check: a smaller target fraction must need more time, and the sign of $\ln(\text{fraction})$ gives the correct positive time. Substitute the answer back into the original exponential, not the logarithmic form, to catch inverted ratios.

Logarithms need dimensionless arguments

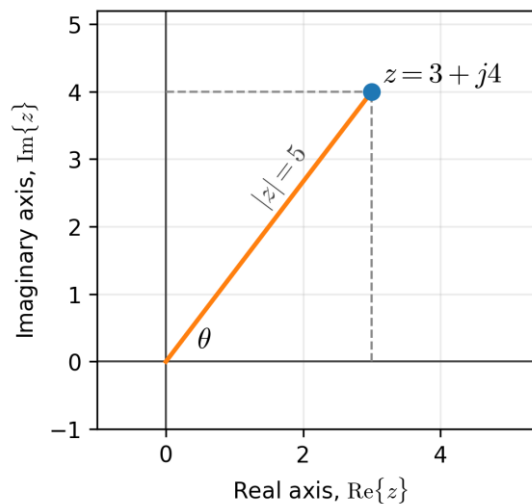
Strictly, the argument of a logarithm must be dimensionless. Expressions such as $\ln(V/V_0)$ are meaningful because the ratio is dimensionless; writing $\ln(5 \text{ V})$ hides this requirement.

5. Complex numbers in rectangular and polar form

Complex numbers are indispensable in EE: they combine magnitude and phase in one quantity, and second-order characteristic equations often have complex-conjugate roots. The imaginary unit is j , with $j^2 = -1$.

$$z = a + jb$$

$$|z| = \sqrt{a^2 + b^2}, \quad \angle z = \tan^{-1}\left(\frac{b}{a}\right)$$

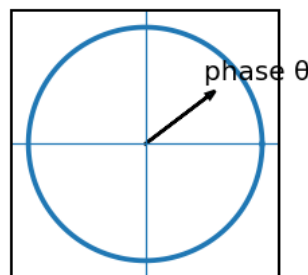


Rectangular and polar forms answer different questions

Rectangular form $z = a + jb$ separates horizontal and vertical components, which suits addition. Polar form $z = r\angle\theta$ separates magnitude and phase, which suits multiplication, division and sinusoidal steady state. Convert deliberately: $r = \sqrt{a^2 + b^2}$, and place θ in the correct quadrant. The conjugate $z^* = a - jb$ reflects the point across the real axis and satisfies $zz^* = |z|^2$. These ideas become essential when characteristic roots are complex.

$$z = a + jb = r(\cos\theta + j\sin\theta) = re^{j\theta}, \quad z^* = a - jb$$

Quadrant discipline: $\tan^{-1}(b/a)$ alone can return an angle in the wrong quadrant when $a < 0$. Use an atan2-style calculation, or inspect the signs of a and b explicitly. This is critical when root angles or phasor phases are converted back to a physical sinusoid.



Worked example — rectangular to polar

Convert $z = 3 + j4$ to polar form.

$$|z| = \sqrt{3^2 + 4^2} = 5, \quad \theta = \tan^{-1}(4/3) = 53.13^\circ$$

$$z = 5\angle 53.13^\circ$$

Result: $3 + j4$ and $5\angle 53.13^\circ$ are the same number in two coordinate systems.

Check: magnitude and quadrant must both agree with the rectangular form. The signs of the real and imaginary parts tell you the quadrant before you touch a calculator.

6. Euler's identity: sinusoidal and exponential forms

Euler's identity links the exponential and trigonometric descriptions of oscillation. It represents a sine–cosine pair compactly and explains why complex roots lead to real damped sinusoids.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Rotation and sinusoids

$e^{j\theta}$ has unit magnitude and phase θ . As θ increases it moves around the unit circle; its real projection is $\cos\theta$ and its imaginary projection is $\sin\theta$. Multiplying by $e^{j\phi}$ adds phase; multiplying by a positive scalar changes amplitude. So a sinusoid can be handled as a complex exponential during the algebra and converted back to a real waveform at the end, preserving amplitude and phase without repeated trigonometric identities.

In ODEs this matters because $e^{j\omega t}$ keeps its frequency under differentiation: each derivative simply contributes a factor $j\omega$. That is the algebraic reason sinusoidal trial functions are so efficient.

Worked example — recovering a cosine

Show that the real part of $6e^{j(200t+\pi/3)}$ is a sinusoid.

$$6e^{j(200t+\pi/3)} = 6\cos(200t + \pi/3) + j 6\sin(200t + \pi/3)$$

$$\operatorname{Re}\{6e^{j(200t+\pi/3)}\} = 6\cos(200t + \pi/3)$$

Result: amplitude 6, angular frequency 200 rad/s, phase $\pi/3$.

Interpretation: the complex exponential is a temporary representation; the physical signal is its real part. Complex notation simplifies the algebra, but the measured waveform is real.

Connection to ODE roots

$$e^{(\sigma+j\omega)t} = e^{\sigma t}[\cos\omega t + j\sin\omega t]$$

The real part σ controls exponential growth or decay; the imaginary part ω controls oscillation. This one observation later makes the shape of second-order responses almost obvious from their roots.

7. Quadratic equations and complex-conjugate roots

Second-order constant-coefficient ODEs reduce to a quadratic characteristic equation. The discriminant tells us whether the roots are two real values, one repeated real value, or a complex-conjugate pair.

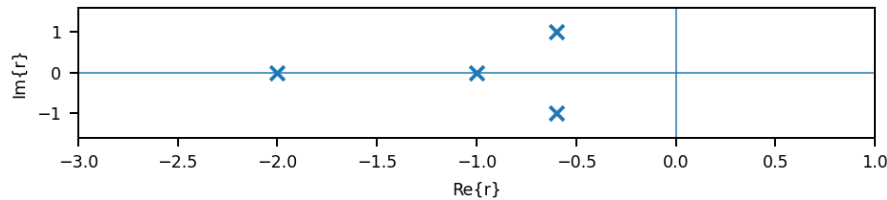
$$ar^2 + br + c = 0 \quad \Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant $\Delta = b^2 - 4ac$	Roots	Response
$\Delta > 0$	two distinct real roots	two exponentials
$\Delta = 0$	repeated real root	$(A + Bt)e^{rt}$
$\Delta < 0$	complex conjugates	damped sinusoid

Read the discriminant first

With real a, b and c , a negative Δ always gives a complex-conjugate pair. The pairing is not optional: it is what keeps the final time response real even though complex arithmetic is used internally. Reading Δ first tells you whether to expect two exponentials, a repeated-root term or a damped sinusoid.

Always substitute the roots back into the quadratic, especially after a negative discriminant. A sign error in the real part changes the damping; a missing conjugate partner would make the final solution complex-valued.



Worked example — complex roots

Solve $r^2 + 4r + 13 = 0$.

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm j6}{2} = -2 \pm j3$$

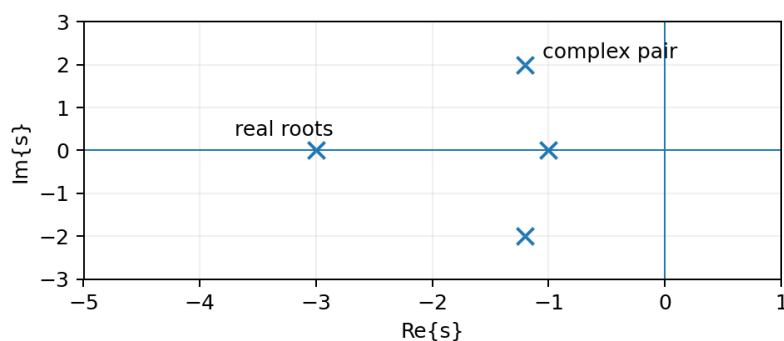
Result: the natural response has an e^{-2t} envelope and oscillates at 3 rad/s.

Reality condition: physical voltages and currents are real, so complex roots come in conjugate pairs whose combined response is real. Read a conjugate pair as "decay or growth plus oscillation": the real part sets the envelope and the imaginary part the angular frequency.

8. Polynomials and characteristic-equation factorisation

A characteristic polynomial encodes the natural modes of a linear constant-coefficient ODE. Factorising exposes the roots, and each root contributes a mode e^{rt} .

$$p(r) = r^2 + 5r + 6 = (r + 2)(r + 3)$$



Factorisation turns an ODE into its natural modes

The characteristic polynomial stores the differential operator in algebraic form. Multiplicity matters: a root repeated twice needs two independent solutions, e^{rt} and te^{rt} (and higher multiplicity brings $t^2 e^{rt}, t^3 e^{rt}, \dots$). For real-coefficient second-order equations, complex roots $\alpha \pm j\beta$ combine into the real form $e^{\alpha t}[C_1 \cos \beta t + C_2 \sin \beta t]$. This is the bridge from polynomial algebra to transient shape.

$$\text{Distinct roots: } (r - r_1)(r - r_2) = 0 \Rightarrow y_n = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Worked example — roots to natural response

For $y'' + 5y' + 6y = 0$, find the general natural solution.

$$r^2 + 5r + 6 = 0 \Rightarrow (r + 2)(r + 3) = 0 \Rightarrow r_1 = -2, r_2 = -3$$

$$y_h(t) = Ae^{-2t} + Be^{-3t}$$

Result: two negative real roots give a non-oscillatory decaying response.

Check: the factor $(r + 3)$ means the root is -3 , giving e^{-3t} , not e^{+3t} . Expand the factors to recover the original polynomial; this reverse check prevents many root errors.

9. From roots to response shape

A root is not merely a number from a polynomial: it defines one natural mode. Its real part controls envelope growth or decay; its imaginary part controls oscillation.

Root pattern	Representative mode	Qualitative behaviour
$r = -a$	e^{-at}	monotonic decay
$r = +a$	e^{at}	growth (instability)
$r = -a \pm j\omega$	$e^{-at}[A\cos\omega t + B\sin\omega t]$	damped oscillation
$r = \pm j\omega$	$A\cos\omega t + B\sin\omega t$	undamped oscillation

Root location is a response map

A root further left in the complex plane decays faster. Roots on the imaginary axis oscillate without decay in the ideal model; real roots give non-oscillatory exponentials; complex-conjugate roots give a sinusoid inside an exponential envelope. Left-half-plane roots imply decaying modes, right-half-plane roots imply growth, and distance from the real axis sets the oscillation frequency. Your later time-domain expression must agree with this prediction.

$$r = \alpha \pm j\beta \Rightarrow \text{envelope } e^{\alpha t}, \text{ angular frequency } |\beta| \text{ rad/s.}$$

Worked example — read the roots before solving

A system has roots $-50 \pm j200 \text{ s}^{-1}$. Find the envelope time constant and the oscillation frequency.

$$\tau = \frac{1}{50} = 0.020 \text{ s}, \quad f_d = \frac{200}{2\pi} = 31.8 \text{ Hz}$$

Result: the response rings at about 31.8 Hz inside an envelope that decays with a 20 ms time constant.

Engineering habit: inspect roots before calculating constants. Stability, oscillation and approximate settling are visible immediately; the constants change only the mixture of modes, not the root locations or the response class.

10. Consolidation and degree-level checkpoint

This chapter supplies the algebraic vocabulary that later ODE methods assume. You should now move freely between exponential, logarithmic, complex and polynomial descriptions.

Essential links

Mathematical object	ODE meaning
e^{rt}	natural mode associated with root r
$\ln(\cdot)$	solves for time in an exponential relation
$\sigma + j\omega$	decay or growth rate plus oscillation rate
quadratic discriminant	classifies second-order roots
polynomial factor	exposes a characteristic root

Combine the algebra checks

A reliable workflow mixes algebra, units and geometry. For a polynomial: identify its order and coefficients, inspect the discriminant or factorisation, locate the roots, and translate each root into a time-domain mode. For exponentials: check that the arguments are dimensionless. For complex numbers: use rectangular form to add and polar form to multiply.

Do not treat roots as the end of the calculation. A root of -20 s^{-1} means a 0.05 s decay time scale; a pair $-3 \pm j40 \text{ s}^{-1}$ means an e^{-3t} envelope with oscillation near 40 rad/s. The numbers should tell a physical story.

Final check: roots must satisfy the original polynomial, and the corresponding modes must satisfy the homogeneous ODE.

Worked example — combined checkpoint

For $r^2 + 20r + 500 = 0$, find the roots and the qualitative response.

$$r = \frac{-20 \pm \sqrt{400 - 2000}}{2} = -10 \pm j20$$

$$y_h(t) = e^{-10t}(A\cos 20t + B\sin 20t)$$

Result: a stable damped oscillation with a 0.1 s envelope time constant and damped angular frequency 20 rad/s.

Interpretation: show the chain explicitly: algebraic form $\rightarrow$ roots $\rightarrow$ response form $\rightarrow$ physical meaning. A final expression without interpretation is less useful than a root-based explanation of damping, oscillation and time scale.

Teach-back questions

1. Why does a passive first-order transient usually contain $e^{-t/\tau}$?
2. Why must logarithm arguments be dimensionless?
3. What do the real and imaginary parts of a complex root control?
4. How does the discriminant classify a second-order response?
5. Why do complex roots come in conjugate pairs for real-coefficient ODEs?

Next chapter: Differentiation and integration techniques turn these algebraic forms into rates of change, accumulated quantities and initial-value calculations.