

Mathematical Language and Notation for ODEs

Ordinary Differential Equations for Electrical Engineers — Foundation Ch. 1 of 15

1. How to read the language of differential equations

Ordinary differential equations become much easier once the notation stops feeling like a code. This chapter does not yet solve difficult circuit equations. Its purpose is to make the language familiar, so that when circuit modelling begins your attention can stay on the engineering.

By the end of the chapter you should be able to: identify dependent and independent variables; distinguish variables from parameters and constants; read first and higher derivatives; explain what integrals and initial conditions contribute; recognise the common EE input functions; classify an equation by order and linearity; check dimensions; and translate a compact ODE into plain language.

A typical electrical-engineering ODE looks like this:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv_s}{dt}$$

Do not read it as one indivisible expression. Read it in layers:

- $i(t)$ is the dependent variable: the unknown current, a function of time.
- t is the independent variable: the quantity with respect to which the current changes.
- L , R and C are parameters describing the circuit components.
- $\frac{di}{dt}$ and $\frac{d^2 i}{dt^2}$ are the first and second rates of change of current.
- $v_s(t)$ is the applied input (forcing function); its derivative appears on the right-hand side.
- The highest derivative is second order, so this is a second-order ODE.

Four questions to ask whenever you meet an ODE

1. What is the unknown function, and what does it depend on?
2. What is the highest derivative, and therefore the order?
3. Which quantities are fixed parameters, and which are inputs or initial conditions?
4. Do the units and limiting behaviour make physical sense?

Central idea: An ODE is not merely mathematics. In engineering it is a compact statement of how a physical system is allowed to evolve.

Reading an ODE in layers

Start with the grammar before doing any algebra. The unknown is a function, such as $i(t)$ or $v(t)$; the independent variable says what it changes with; coefficients such as R , L and C describe the system; and the right-hand side describes what drives it. A derivative is not a separate unknown: di/dt and d^2i/dt^2 are rates of change of the same current $i(t)$. Read aloud in this order, the symbols become a physical statement rather than a wall of notation.

A useful translation pattern: storage terms + dissipation terms + static terms = forcing.

Worked example — read an ODE before solving it

For $L i'' + R i' + \frac{1}{C} i = v_s'(t)$, the dependent variable is $i(t)$ and the independent variable is t . The highest derivative is i'' , so the equation is second order. L , R and C are parameters; $v_s(t)$ is the prescribed source.

Unit check: every term has units of V/s in this differentiated form, so the model passes the first structural check.

2. Functions, variables, parameters and graphs

A function assigns an output value to each allowed input value. In most lumped-circuit ODEs, time is the independent variable and voltage or current is the dependent variable, so we write $v(t)$, $i(t)$, $q(t)$ and $\phi(t)$. The parentheses matter: v names a voltage, while $v(t)$ emphasises that its value may change with time.

Symbol	Role	Typical EE meaning
t	independent variable	time, s
$v(t)$	dependent variable	voltage, V
$i(t)$	dependent variable	current, A
R, L, C	parameters	resistance, inductance, capacitance
$v_s(t)$	input / forcing function	applied source voltage
$v(0^+)$	initial value	voltage immediately after switching

Independent and dependent variables

If $i = i(t)$, the current may change when t changes, and di/dt asks how rapidly it does so. An ordinary differential equation has one independent variable. A field quantity such as $E(x, y, z, t)$ depends on several independent variables and leads naturally to partial differential equations.

Variables versus parameters

A resistor value R may appear in an ODE, but it is normally a fixed parameter during an analysis. The current $i(t)$ is different: it is the unknown function to be solved. Parameters shape the response without themselves being functions of time in the simplest model.

$$RC \frac{dv}{dt} + v = v_s(t)$$

Here $v(t)$ is unknown, t is the independent variable, R and C are parameters, and $v_s(t)$ is a prescribed input. Changing R or C changes the time scale of the response; changing $v_s(t)$ changes the forcing.

Graphs are part of the mathematics

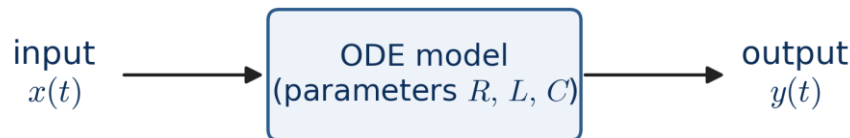
An ODE solution is a whole curve, not one number, so a graph is often as informative as the formula. Label the horizontal axis with the independent variable (usually time) and the vertical axis with the solved quantity and its unit. Mark the initial value, final value, time constants, peaks and oscillation periods where relevant. Domain and range matter too: a model may be valid only for $t \geq 0$ or only within a specified operating region.

Notation habit: Whenever you write $y(t)$, ask what y physically represents, what its unit is, and what event defines $t = 0$. These three questions prevent many later mistakes.

Parameter, input and output

Parameters such as τ , R or C stay fixed while t changes. An input $x(t)$ may also vary with time, but it is prescribed; the output $y(t)$ is what we must determine.

Think: parameter = setting of the system; input = imposed function; output (state) = unknown function.



Worked example — separate variable, parameter and input

In $\tau y' + y = Kx(t)$, $y(t)$ is the unknown output, $x(t)$ is the input, τ is a time parameter and K is a gain. If $x(t) = 3$ for $t \geq 0$, the input is known even though its switching description depends on time.

At steady DC, $y' = 0$, so $y = 3K$. This prediction is available before solving for the transient.

3. Derivatives as rates of change

A derivative measures how rapidly a quantity changes. In electrical engineering this is not abstract: derivatives appear directly in the capacitor and inductor laws, and hence in the ODEs of every dynamic circuit.

$$i_C(t) = C \frac{dv_C}{dt}$$

$$v_L(t) = L \frac{di_L}{dt}$$

Capacitor current is proportional to the rate of change of capacitor voltage; inductor voltage is proportional to the rate of change of inductor current. If the stored-energy variable is not changing, the corresponding derivative is zero.

First, second and higher derivatives

Notation	Read aloud	Interpretation
$\frac{dy}{dt}$ or y'	first derivative of y	rate of change
$\frac{d^2y}{dt^2}$ or y''	second derivative of y	rate of change of the rate of change
$\frac{d^ny}{dt^n}$	n th derivative of y	higher-order dynamic behaviour

For a voltage $v(t)$, dv/dt has units V/s and d^2v/dt^2 has units V/s². Differentiation changes the dimensions of a quantity, which makes unit checking especially valuable in ODE work.

Derivative of an exponential

The exponential is special because differentiating it reproduces the same function:

$$\frac{d}{dt}[Ae^{st}] = sAe^{st}$$

This is why exponentials dominate the natural responses of linear constant-coefficient systems. It is also why Laplace transforms work so well: differentiation becomes multiplication by the complex variable s .

Local slope and sign

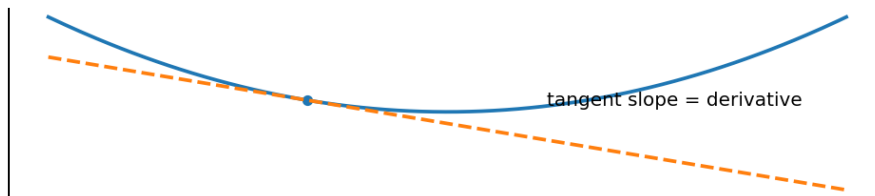
If $dv/dt > 0$ the voltage is increasing at that instant; if $dv/dt < 0$ it is decreasing. A zero derivative means zero local slope, not a zero quantity. At a local maximum or minimum the slope is zero although the function is not. In steady DC, an ideal capacitor voltage may be non-zero while $dv/dt = 0$.

Common trap: Do not confuse "the derivative is zero" with "the quantity is zero". A constant 5 V signal has $dv/dt = 0$ but $v = 5$ V.

What a derivative predicts

A first derivative says how fast a quantity is changing now; a second derivative says how that rate itself is changing. A large value of y does not imply a large derivative, and a zero derivative does not imply that y is zero.

For a small interval Δt : $y(t + \Delta t) \approx y(t) + y'(t) \Delta t$. The derivative is the coefficient that predicts the local change.



Worked example — read slope from a waveform

Let $v(t) = 5 - 2t + t^2$ V. Then $v'(t) = -2 + 2t$ V/s and $v''(t) = 2$ V/s². At $t = 1$ s, $v' = 0$ even though $v = 4$ V.

Interpretation: the voltage is momentarily flat at $t = 1$ s, and the positive second derivative shows the curve bending upward.

4. Integrals, accumulation and initial values

Integration reverses differentiation and describes accumulation. A derivative gives the instantaneous rate of change; an integral gives the accumulated change over an interval.

$$v_C(t) = v_C(t_0) + \frac{1}{C} \int_{t_0}^t i_C(\tau) d\tau$$

The capacitor voltage at time t equals its known voltage at the start time t_0 plus the accumulated effect of the current flowing into it. The symbol τ is a dummy integration variable; it avoids confusion with the upper limit t .

Constants of integration

An indefinite integral produces a family of functions, because differentiation removes constants. For example:

$$\frac{dy}{dt} = a \quad \Rightarrow \quad y(t) = at + K$$

The constant K cannot be found from the differential equation alone. An initial or boundary condition is needed.

Initial conditions select one physical trajectory

Suppose $y(0) = y_0$. Substituting $t = 0$ gives $K = y_0$, so $y(t) = at + y_0$. The ODE defines a family of allowed trajectories; the initial condition picks out the one the real system follows.

A first-order ODE normally needs one independent initial condition; a second-order ODE needs two. In circuits these usually come from stored energy: capacitor voltage and inductor current.

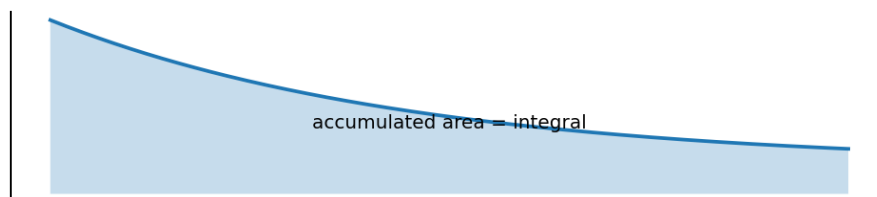
Definite versus indefinite integration

Form	Meaning
$\int f(t) dt$	family of antiderivatives; include an arbitrary constant
$\int_{t_0}^t f(\tau) d\tau$	accumulated change from a specified start time; the initial value appears explicitly

Definite integrals are often clearer in engineering: the new state equals the old state plus the accumulated rate of change. The units are transparent too, since integrating a rate over time returns the original quantity.

Physical interpretation: Initial conditions are not decoration. They carry the memory of energy already stored in the system when the analysis begins.

$y(t) = y(t_0) + \int_{t_0}^t y'(\lambda) d\lambda$. The dummy variable λ avoids confusion with the upper limit t .



Worked example — recover a state from its rate

Suppose $i(t) = 2e^{-t}$ A flows into a capacitor and $q(0) = 1$ C. Then

$$q(t) = 1 + \int_0^t 2e^{-\lambda} d\lambda = 1 + 2(1 - e^{-t}) \text{ C}$$

Check: differentiating $q(t)$ returns $i(t)$, and $q(0) = 1$ C. The initial charge has fixed the constant.

5. Algebra, units and normalised ODE form

Before solving an ODE it often helps to rearrange it into standard form. This exposes the coefficients that control the response and makes dimensional errors easier to spot. Consider the first-order RC equation:

$$RC \frac{dv}{dt} + v = v_s(t)$$

Dividing every term by RC gives

$$\frac{dv}{dt} + \frac{1}{RC} v = \frac{1}{RC} v_s(t)$$

The coefficient $1/(RC)$ must have units of inverse time. Since R is in Ω and C in F, RC is in seconds, so we define the time constant $\tau = RC$ and write

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{1}{\tau} v_s(t)$$

Dimensional consistency

Every additive term in a physical equation must have the same dimensions. In the original RC equation, $RC dv/dt$ has units $s \cdot V/s = V$, matching both v and v_s . In the normalised form every term has units V/s . This is an excellent error check: if added terms do not match, a coefficient, derivative or unit conversion is missing.

Normalisation reveals time scales

Standard form makes the natural time scale obvious. For a first-order system, the coefficient of the dependent variable (after normalising the highest derivative) often contains $1/\tau$. For second-order systems, the normalised coefficients have units s^{-1} and s^{-2} and become damping and natural-frequency parameters.

A repeatable algebra check

1. Choose the dependent variable explicitly.
2. Collect like derivative terms, in descending derivative order.
3. Divide by the coefficient of the highest derivative if convenient.
4. Check the units of every term after rearrangement.
5. Identify coefficients with dimensions of inverse time or inverse time squared.

Key idea: Rearrangement may change how an ODE looks, but it must not change its physical meaning.

$$\text{Pattern: } a_1 y' + a_0 y = f(t) \Rightarrow y' + \frac{a_0}{a_1} y = \frac{f(t)}{a_1}$$

Worked example — normalise and check units

From $0.02 y' + y = 4u(t)$, divide every term by 0.02 to obtain $y' + 50y = 200u(t)$. The coefficient 50 has units s^{-1} .

Time scale: $\tau = 1/50 = 0.020$ s. Reading the coefficient already tells us how fast the transient will be.

6. Exponential functions and natural response

The exponential is central to ODEs because differentiation reproduces it. Suppose a source-free first-order system satisfies

$$\frac{dy}{dt} + ay = 0$$

Try $y = Ke^{st}$. Substitution gives

$$sKe^{st} + aKe^{st} = 0$$

For a non-zero solution we can divide out Ke^{st} , leaving $s + a = 0$. Therefore $s = -a$ and

$$y(t) = Ke^{-at}$$

If $a > 0$ the response decays to zero; if $a < 0$ it grows without bound. In passive first-order circuits the natural response normally decays, because resistance dissipates the stored energy.

Time-constant form

With $a = 1/\tau$ the response becomes

$$y(t) = Ke^{-t/\tau}$$

Time	$e^{-t/\tau}$	Fraction of initial transient remaining
0	1.000	100%
τ	0.368	36.8%
2τ	0.135	13.5%
3τ	0.0498	5.0%
5τ	0.00674	0.67%

This is why engineers treat a first-order transient as practically settled after about five time constants.

Growth, decay and sign

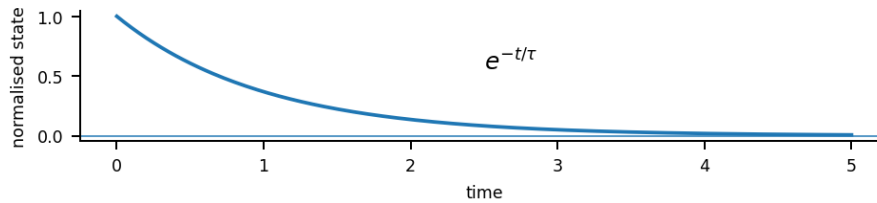
The sign of the exponent matters: $e^{-t/\tau}$ decays for positive τ , while $e^{+t/\tau}$ grows. A sign error in the characteristic root can turn a stable passive circuit into an impossible growing response.

Dimensional habit: Whenever an exponential appears, inspect the exponent first: it must be dimensionless, so the coefficient multiplying t must have units s^{-1} .

Why exponentials reproduce themselves

If $y = e^{st}$ then $y' = se^{st}$: differentiation changes only a scale factor, not the time dependence. Linear constant-coefficient ODEs therefore turn differentiation into multiplication by s , which is why exponential trial functions work so well. The sign of $\text{Re}\{s\}$ controls growth or decay; in a passive first-order circuit we expect a negative real exponent.

If $y(t) = Ae^{-t/\tau}$, then $y' = -\frac{1}{\tau}y$. After one time constant, y has fallen to $e^{-1} \approx 0.368$ of its initial value.



Worked example — compare two time constants

For $y_1 = e^{-t/0.1}$ and $y_2 = e^{-t/0.5}$, both start at 1. At $t = 0.1$ s, $y_1 = e^{-1} = 0.368$ while $y_2 = e^{-0.2} = 0.819$.

The smaller time constant decays faster, because the exponent coefficient $1/\tau$ is larger.

7. Sinusoids, complex exponentials and common EE inputs

Electrical systems are driven by more than DC steps. Sinusoids, exponentials, ramps and pulses all appear in ODE problems, and it helps to recognise their forms before learning solution techniques.

Input type	Typical notation	Key feature
constant / DC	A	unchanging after application
step	$A u(t)$	switches from 0 to A at $t = 0$
ramp	$A t u(t)$	increases linearly with time
exponential	$A e^{at}$	growth or decay
sinusoid	$A \cos(\omega t + \phi)$	periodic forcing
pulse	$A[u(t) - u(t - T)]$	finite-duration step pair

Sinusoidal functions

$$x(t) = A \cos(\omega t + \phi)$$

A is the amplitude, ω the angular frequency in rad/s, and ϕ the phase in radians. Differentiation changes amplitude and phase but preserves the sinusoidal form. That is why linear circuits driven sinusoidally have sinusoidal steady-state responses at the same frequency.

Complex exponential representation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

A real sinusoid is the real part of a complex exponential. This is the basis of phasors, and it connects directly to Laplace methods, where $s = \sigma + j\omega$ combines exponential growth or decay with oscillation:

$$e^{st} = e^{\sigma t} e^{j\omega t}, \quad s = \sigma + j\omega$$

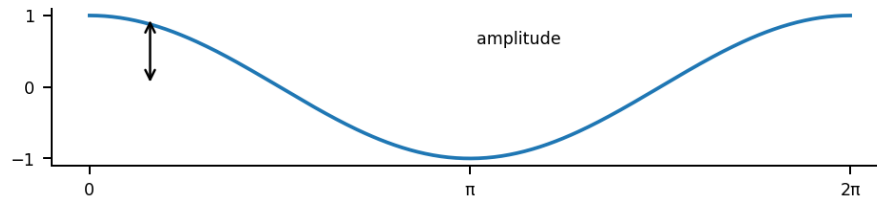
If $\sigma < 0$ the envelope decays; if $\sigma = 0$ the sinusoid is sustained; if $\sigma > 0$ the envelope grows. Complex-conjugate values of s therefore produce damped sinusoids in real systems.

EE connection: Phasor analysis gives the sinusoidal steady state. ODE and Laplace analysis give the complete response, including the transient immediately after switching.

Two views of the same signal

Euler's identity packages cosine and sine into one exponential. The complex form is an analysis tool; the measured voltage or current is the real waveform. Frequency sets how fast the phase advances, while amplitude and phase set the scale and horizontal shift. Complex roots of ODEs later produce damped sinusoids because $e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t}$.

$$\cos(\omega t + \phi) = \operatorname{Re}\{e^{j(\omega t + \phi)}\} \text{ and } e^{(\alpha+j\omega)t} = e^{\alpha t}[\cos\omega t + j\sin\omega t]$$



Worked example — convert a damped complex mode

Take $e^{(-2+j5)t}$. Its real part is $e^{-2t}\cos 5t$: the factor e^{-2t} is the decaying envelope and 5 rad/s is the oscillation frequency.

This is exactly the form that appears when a real second-order ODE has roots $-2 \pm j5$.

8. Order, degree, linearity and homogeneity

Classify an ODE before solving it. The classification tells you which methods are available and what kind of response to expect.

Order

The order is the order of the highest derivative present.

Equation	Order
$\frac{dy}{dt} + ay = f(t)$	first
$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t)$	second
$\frac{d^3y}{dt^3} + ay = 0$	third

Degree

When an equation is polynomial in its derivatives, the degree is the power of the highest-order derivative, after removing fractions and radicals involving derivatives. Most circuit ODEs in a first EE course have degree one.

Linear versus nonlinear

A linear ODE contains the dependent variable and its derivatives only to the first power and never multiplied together. Coefficients may depend on the independent variable, but not on the dependent variable.

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Terms such as y^2 , $y y'$, $\sin y$ or $(y')^2$ make an equation nonlinear. Ideal RLC networks with constant R , L and C are linear; circuits with strongly nonlinear device laws may not be.

Homogeneous versus non-homogeneous

A linear ODE is homogeneous when the forcing term is zero. Its solution describes the natural response. A non-homogeneous equation has non-zero forcing and also needs a forced (particular) response.

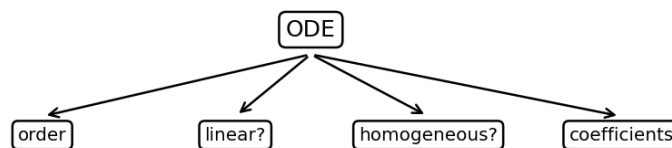
$$y' + ay = 0 \quad \text{versus} \quad y' + ay = f(t)$$

Common trap: "Homogeneous" does not mean a zero solution. A homogeneous ODE has a non-zero natural response whenever the initial conditions are non-zero.

Classification tells you which methods are legal

Order is set by the highest derivative. Degree applies only when the equation is polynomial in its derivatives. Linearity is stricter: y and its derivatives appear only to the first power, do not multiply one another, and have coefficients that may depend on t but not on y . "Homogeneous" and "non-homogeneous" apply once the equation is in linear operator form. A nonlinear equation can be first order, and a linear equation can have time-varying coefficients.

Linear template: $a_n(t)y^{(n)} + \dots + a_1(t)y' + a_0(t)y = f(t)$. Homogeneous means $f(t) = 0$.



Worked example — classify three equations

$y' + 3y = \sin t$ is first order, linear and non-homogeneous. $y' + 3y^2 = 0$ is first order but nonlinear. $y'' + (1 + t)y = 0$ is second order, linear and homogeneous, with a time-varying coefficient.

Classification depends on structure, not on whether the equation looks difficult.

9. Reading complete ODE models without getting lost

The goal now is to translate compact equations into structured engineering statements. This skill is worth more than memorised solution formulas.

Example 1: first-order circuit form

$$\tau \frac{dy}{dt} + y = Kx(t)$$

1. $y(t)$ is the output or state variable being solved.
2. $x(t)$ is a prescribed input.
3. τ has units of time and sets the transient time scale.
4. K is a gain relating the steady output to the input.
5. The equation is first order because dy/dt is the highest derivative.

Example 2: second-order constant-coefficient form

$$y'' + 2\alpha y' + \omega_0^2 y = f(t)$$

Read it as: the acceleration-like term y'' , the damping term $2\alpha y'$ and the restoring term $\omega_0^2 y$ together balance the forcing $f(t)$. In an RLC circuit, α and ω_0 are combinations of R , L and C . Their relative size decides whether the natural response is overdamped, critically damped or underdamped.

Example 3: initial-value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

The differential equation defines a family of second-order natural responses; the two initial conditions select one member. Because the equation is second order, two independent initial conditions are required.

A reading sequence that scales to harder problems

1. Name the unknown function and its physical unit.
2. Identify the independent variable.
3. Find the highest derivative and classify the order.
4. Separate coefficients and parameters from the forcing function.
5. Identify the initial conditions and the switching time reference.
6. Check dimensions and ask what the solution should do as $t \rightarrow \infty$.

Teaching principle: A correct solution method begins with a correct reading of the model. Most later circuit errors trace back to a sign convention, an initial condition or a misidentified variable.

Read the model before solving it

First identify what each term would do if it acted alone. A derivative term usually represents storage or inertia; an undifferentiated state term often represents a restoring or dissipative effect; the forcing term is the external drive. Then locate the natural time scale. In $\tau y' + y = Kx(t)$, τ sets how quickly the transient changes and K sets the long-term input–output scaling. This lets you predict the shape of the solution before calculating any constants.

At DC the derivatives vanish, so $\tau y' + y = Kx_0$ gives $y(\infty) = Kx_0$ when the system is stable.

Worked example — predict the response before solving

For $0.1 y' + y = 2u(t)$ with $y(0) = 0$, the equation is first order and stable. The time constant is 0.1 s and the DC final value is 2.

So the response starts at 0 and approaches 2, and the ODE gives an initial slope $y'(0) = 20$ units/s.

10. Consolidation, teach-back and degree-level checkpoint

This chapter has succeeded if you can now parse the notation of an ODE before you know the full solution method.

Essential distinctions

Pair	Distinction
t vs $y(t)$	independent variable versus dependent function
R, L, C vs $v(t), i(t)$	fixed model parameters versus time-varying quantities
y' vs y''	first rate of change versus second rate of change
ODE vs PDE	one independent variable versus several
homogeneous vs forced	zero forcing versus non-zero forcing
transient vs steady state	decaying, temporary component versus long-time behaviour
initial condition vs input	stored state at the start versus externally applied excitation

Mini worked checks

- Order.** In $3y'' + 2y' + y = 5u(t)$ the highest derivative is y'' , so the equation is second order.
- Units.** If $\tau dy/dt$ and y are added, τ must have units of time.
- Initial conditions.** A first-order ODE normally needs one independent initial value; a second-order ODE needs two.
- Exponential.** In $e^{-t/\tau}$, the ratio t/τ is dimensionless.
- Linearity.** $y' + 3y = \sin t$ is linear; $y' + 3y^2 = \sin t$ is nonlinear.

Teach-back questions

- Why do capacitors and inductors lead naturally to differential equations?
- What is the difference between a parameter and a dependent variable?
- Why does a second-order ODE need two initial conditions?
- Why do exponentials appear so often in natural responses?
- What is the physical difference between a homogeneous and a forced equation?
- How can units reveal a missing coefficient or derivative?
- What does the sign of an exponent tell you?
- Why is phasor steady-state analysis not, by itself, a complete transient solution?

Degree-level checkpoint

You are ready for the next chapter if you can look at

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{dv_s}{dt}$$

and say confidently that $i(t)$ is the dependent variable and t the independent variable; the equation is linear and second order; L , R and C are circuit parameters; $v_s(t)$ is the forcing input; two independent initial conditions are needed; every term must have consistent units; and the solution will contain both natural and forced behaviour when the source is non-zero.

A repeatable pre-solution checklist

Most lost marks come from starting a method before classifying the equation. Every time: name the unknown and its units; identify the independent variable; find the highest derivative; classify linearity; isolate the

forcing; count the independent initial conditions; and check dimensions. Then predict the long-term behaviour and any continuity constraints. Only then choose separation, an integrating factor, a characteristic equation or a transform method.

A good final verification has three parts: substitute the solution into the ODE, apply every initial condition, and check the limiting behaviour.

Worked example — full pre-solution checkpoint

$y'' + 6y' + 25y = 10\cos 4t$ is a second-order, linear, non-homogeneous ODE with constant coefficients. Two initial conditions are required, and the forcing is sinusoidal.

The characteristic equation $r^2 + 6r + 25 = 0$ has roots $-3 \pm j4$: a decaying oscillation. Any later detailed solution must agree with this.

Next chapter: Foundation Ch. 2 develops the algebra, exponentials, logarithms, complex numbers and polynomial roots used repeatedly in ODE solutions.