

Worked Examples - Maxwell and Wave Theory

Electromagnetic Theory Course — Teaching, Synthesis and Worked Examples — Document 21 of 23

Ten worked examples reinforcing Documents 5-10: fields, Maxwell's equations, phasors, wave propagation, polarization and power.

Worked Example 1 - Electric field and flux density in a dielectric

A parallel-plate structure contains a dielectric with $\epsilon_r = 4$. The field magnitude is $E = 2.0$ kV/m.



Parallel-plate field

For a linear isotropic dielectric,

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E}.$$

Hence

$$D = (8.854 \times 10^{-12})(4)(2000) = 7.083 \times 10^{-8} \text{ C/m}^2.$$

Therefore

$$\boxed{D = 70.8 \text{ nC/m}^2.}$$

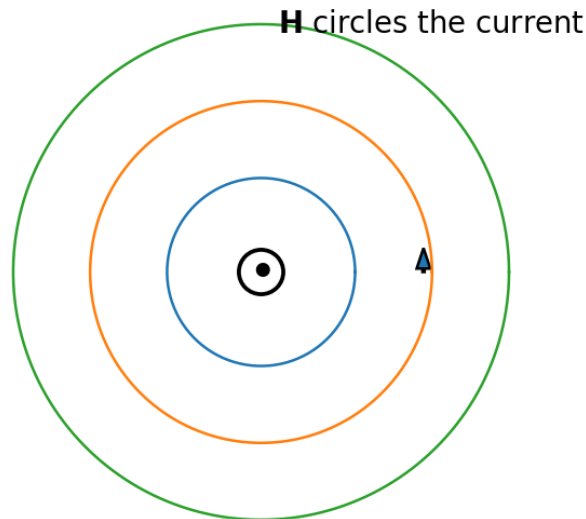
If the plate area is $A = 0.01 \text{ m}^2$, the free charge magnitude is

$$Q = DA = 7.08 \times 10^{-10} \text{ C}.$$

This example illustrates the distinction between $\mathbf{E}$, which describes electric force per unit charge, and $\mathbf{D}$, whose flux is directly related to free charge.

Worked Example 2 - Magnetic field around a long straight conductor

A long conductor carries $I = 5$ A. Find H and B at radius $r = 20$ mm in free space.



Magnetic circulation

Ampere's law for a circular contour gives

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I.$$

By symmetry, H is tangential and constant on the circle:

$$H(2\pi r) = I,$$

so

$$H = \frac{I}{2\pi r} = \frac{5}{2\pi(0.020)} = 39.8 \text{ A/m}.$$

Then

$$B = \mu_0 H = (4\pi \times 10^{-7})(39.8) = 50.0 \text{ } \mu\text{T}.$$

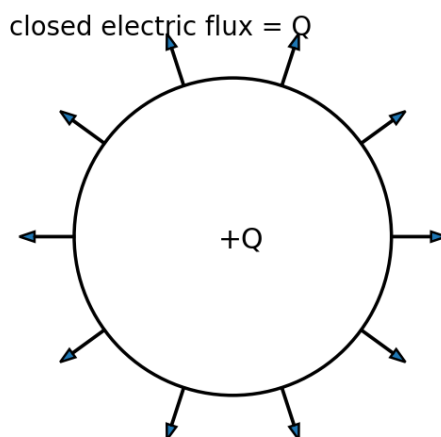
Thus

$$\boxed{H = 39.8 \text{ A/m}, \quad B = 50.0 \text{ } \mu\text{T}.}$$

The circular direction follows the right-hand rule.

Worked Example 3 - Use Gauss's law for a spherical charge

A point charge $Q = 2 \text{ nC}$ sits at the centre of an imaginary sphere of radius $r = 0.10 \text{ m}$.



Gaussian surface

Gauss's law is

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q.$$

Spherical symmetry makes D radial and constant over the sphere:

$$D(4\pi r^2) = Q.$$

Hence

$$D = \frac{Q}{4\pi r^2} = \frac{2 \times 10^{-9}}{4\pi(0.1)^2} = 15.9 \text{ nC/m}^2.$$

In free space,

$$E = \frac{D}{\epsilon_0} = 1.80 \text{ kV/m}.$$

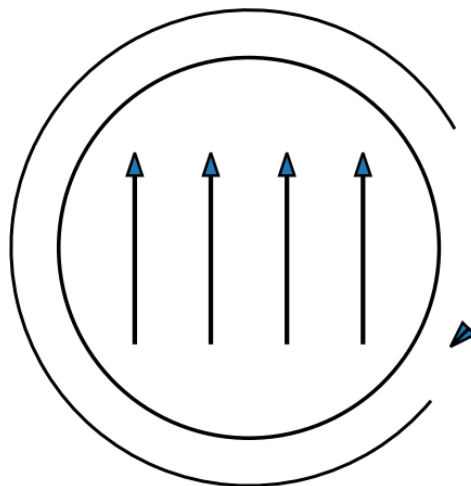
The point is not merely the inverse-square law: **symmetry plus Gauss's law creates the inverse-square law.**

Worked Example 4 - Faraday's law and induced emf

A circular loop encloses an area $A = 0.010 \text{ m}^2$. A uniform normal magnetic flux density increases at

$$\frac{dB}{dt} = 0.50 \text{ T/s}.$$

Changing magnetic flux induces circulating E



Faraday induction

Magnetic flux is

$$\Phi_B = BA,$$

so

$$\frac{d\Phi_B}{dt} = A \frac{dB}{dt} = 0.010(0.50) = 5.0 \times 10^{-3} \text{ Wb/s}.$$

Faraday's law gives

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

Therefore

$$\boxed{\mathcal{E} = -5.0 \text{ mV.}}$$

The minus sign is Lenz's law: the induced circulation acts in the direction that opposes the change in magnetic flux.

Worked Example 5 - Why Maxwell added displacement current

A parallel-plate capacitor has $C = 100$ pF and voltage

$$v(t) = 10 \cos(2\pi \times 10^6 t) \text{ V.}$$

The conduction current in the wire is

$$i(t) = C \frac{dv}{dt}.$$

The peak current is therefore

$$I_{pk} = \omega C V_{pk} = (2\pi \times 10^6)(100 \times 10^{-12})(10) = 6.28 \text{ mA.}$$

There is no conduction current crossing the dielectric gap, yet Ampere's law must give the same magnetic field regardless of which surface spans the contour. Maxwell's correction supplies

$$I_d = \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{S}.$$

For an ideal capacitor,

$$\boxed{I_d = I_c.}$$

That equality is not bookkeeping: it restores charge continuity and makes electromagnetic-wave propagation possible.

Worked Example 6 - Convert a sinusoid to a phasor and differentiate it

Let

$$e(t) = 20\cos(\omega t - 30^\circ) \text{ V/m.}$$

Using the peak-value phasor convention,

$$\tilde{E} = 20\angle(-30^\circ) \text{ V/m.}$$

The time derivative is

$$\frac{de}{dt} = \text{Re}\{j\omega\tilde{E}e^{j\omega t}\}.$$

Thus differentiation in the phasor domain is multiplication by $j\omega$:

$$\boxed{\frac{\partial}{\partial t} \rightarrow j\omega.}$$

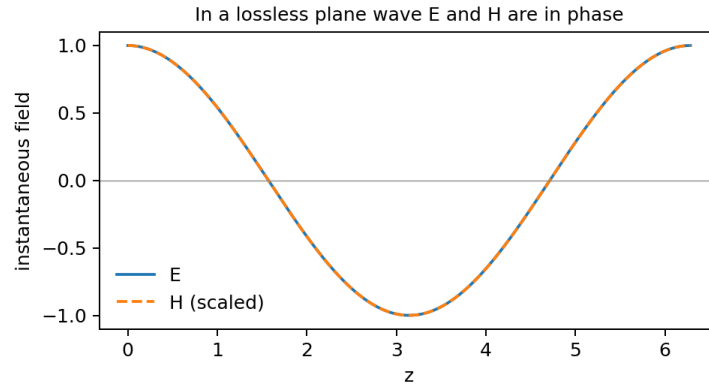
The derivative phasor is

$$j\omega\tilde{E} = 20\omega\angle(60^\circ).$$

Multiplication by j rotates a phasor by +90 degrees. This is the key step that turns Maxwell's time-domain differential equations into compact algebraic-frequency-domain equations.

Worked Example 7 - Plane wave wavelength and E-H relationship

A 3 GHz plane wave propagates in free space.



Plane-wave fields

The wavelength is

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8}{3.00 \times 10^9} = 0.100 \text{ m.}$$

Therefore

$$\beta = \frac{2\pi}{\lambda} = 62.83 \text{ rad/m.}$$

If the electric-field peak amplitude is $E_0 = 10 \text{ V/m}$, the magnetic-field amplitude is

$$H_0 = \frac{E_0}{\eta_0} = \frac{10}{376.73} = 26.5 \text{ mA/m.}$$

Hence a +z travelling wave could be written

$$\begin{aligned}\vec{\mathbf{E}} &= 10\hat{\mathbf{x}}e^{-j\beta z}, \\ \vec{\mathbf{H}} &= 0.0265\hat{\mathbf{y}}e^{-j\beta z}.\end{aligned}$$

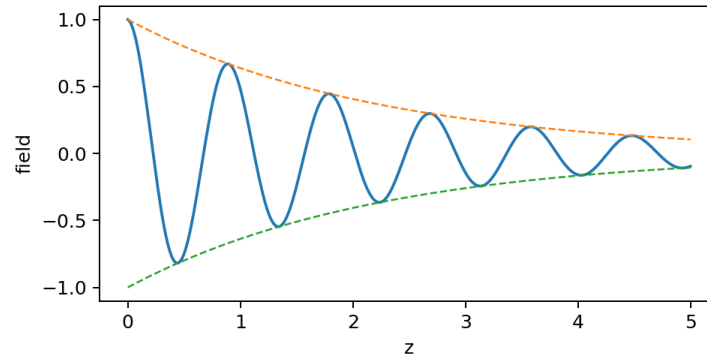
The triplet $\mathbf{E}, \mathbf{H}, \mathbf{k}$ is mutually perpendicular.

Worked Example 8 - Attenuation in a lossy medium

Suppose the propagation constant is

$$\gamma = \alpha + j\beta = 0.40 + j25 \text{ m}^{-1}.$$

Lossy wave: $e^{-\alpha z} \cos(\omega t - \beta z)$



Lossy propagation

A forward wave contains

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}.$$

At $z = 3$ m, the field-amplitude ratio is

$$\frac{|E(3)|}{|E(0)|} = e^{-0.4(3)} = 0.301.$$

In decibels for field amplitude,

$$20 \log_{10}(0.301) = -10.43 \text{ dB}.$$

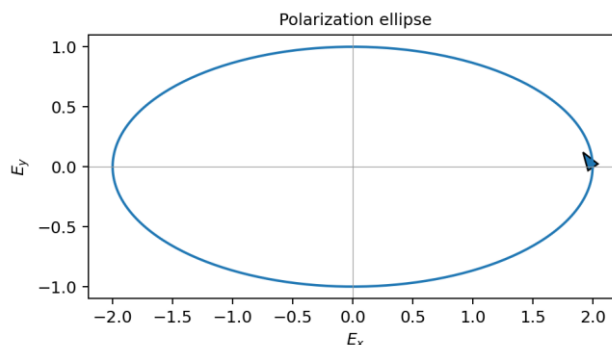
Power is proportional to field magnitude squared, so the power ratio is $0.301^2 = 0.0907$, which also corresponds to -10.43 dB. The same dB number appears because field uses $20 \log$ while power uses $10 \log$.

Worked Example 9 - Determine a polarization state

Let the electric field components be

$$E_x(t) = 4\cos\omega t,$$

$$E_y(t) = 2\cos(\omega t + 90^\circ).$$



Polarization ellipse

Because the components are in quadrature and have unequal amplitudes, the tip of $\mathbf{E}$ traces an ellipse. Eliminating time gives

$$\left(\frac{E_x}{4}\right)^2 + \left(\frac{E_y}{2}\right)^2 = 1.$$

So this is **elliptical polarization**, with axial ratio

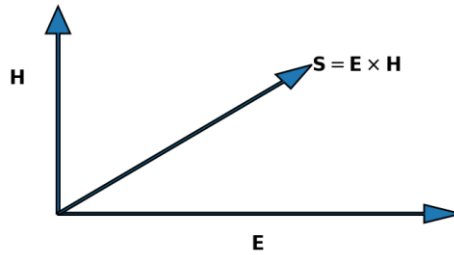
$$AR = \frac{4}{2} = 2.$$

Equal magnitudes with a 90° phase difference would give circular polarization. Zero phase difference would give linear polarization.

The handedness depends on propagation direction and time convention, so always state both before assigning RHCP/LHCP.

Worked Example 10 - Time-average Poynting power through an aperture

A free-space plane wave has RMS electric-field magnitude $E_{rms} = 10$ V/m.



Poynting direction

For free space,

$$H_{rms} = \frac{E_{rms}}{\eta_0}.$$

The average power density is

$$\langle S \rangle = E_{rms} H_{rms} = \frac{E_{rms}^2}{\eta_0} = \frac{100}{376.73} = 0.265 \text{ W/m}^2.$$

If the wave crosses a flat 0.50 m^2 aperture normally,

$$P = \iint_A \langle \mathbf{S} \rangle \cdot d\mathbf{A} = 0.265(0.50) = 0.133 \text{ W}.$$

Therefore

$$\boxed{P \approx 133 \text{ mW.}}$$

Core EM checkpoint. You should now be able to move from sources and Maxwell's equations to sinusoidal waves, propagation, polarization and measurable power without treating the formulas as disconnected facts.