

Worked Examples - Mathematical Foundations

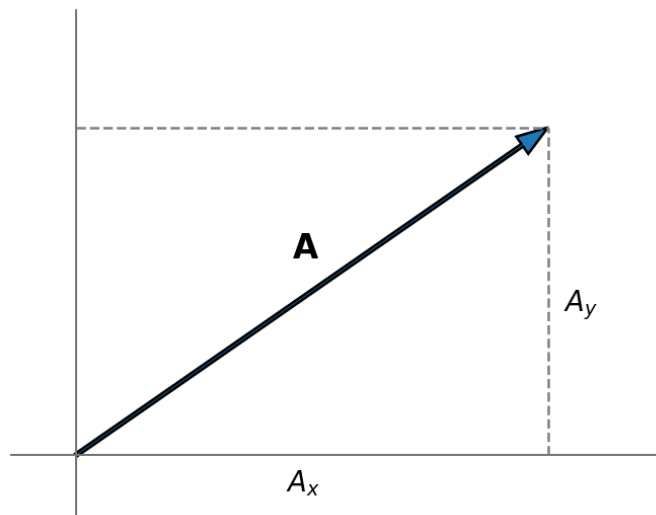
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Ten worked examples reinforcing Documents 1-4: notation, complex numbers, vector algebra, coordinate systems, partial differentiation and integration.

Worked Example 1 - Resolve a field vector into components

Suppose an electric field at one point is

$$\mathbf{E} = 30\hat{x} + 40\hat{y} \text{ V/m.}$$



Vector components

The magnitude is

$$|\mathbf{E}| = \sqrt{30^2 + 40^2} = 50 \text{ V/m.}$$

The direction measured from the +x axis is

$$\theta = \tan^{-1}\left(\frac{40}{30}\right) = 53.13^\circ.$$

So the same vector can be described either by Cartesian components or by magnitude and direction:

$$\boxed{\mathbf{E} = 30\hat{x} + 40\hat{y} \text{ V/m} = 50\angle 53.13^\circ \text{ V/m.}}$$

Be careful: the polar-angle notation is being used here to describe a **real spatial vector**, not a complex phasor. The symbol $\angle$ can appear in both contexts, so context matters.

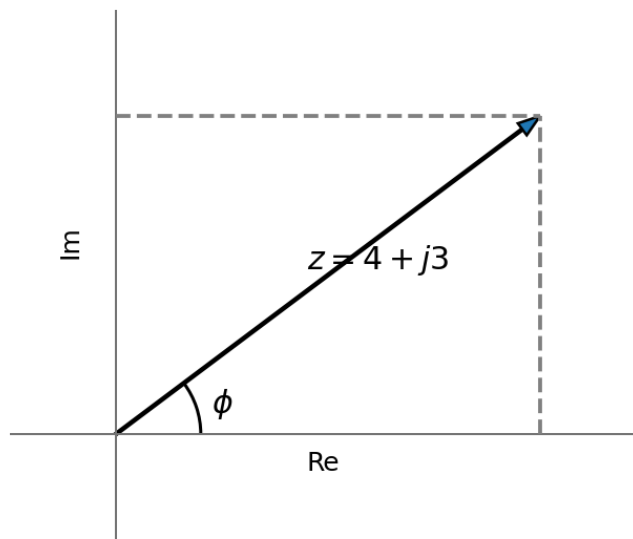
Physical interpretation. A test charge would experience force $\mathbf{F} = q\mathbf{E}$. Both components act simultaneously; the magnitude is not $30 + 40$.

Teach-back: why is $|\mathbf{E}|$ a scalar even though $\mathbf{E}$ is a vector?

Worked Example 2 - Convert a complex RF quantity between forms

Let

$$z = 4 + j3.$$



Argand diagram

Magnitude:

$$|z| = \sqrt{4^2 + 3^2} = 5.$$

Phase:

$$\phi = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ.$$

Therefore

$$\boxed{z = 5\angle 36.87^\circ = 5e^{j0.6435}}.$$

To return to rectangular form, use Euler's identity:

$$5e^{j\phi} = 5(\cos\phi + j\sin\phi) = 4 + j3.$$

The conjugate is

$$z^* = 4 - j3,$$

and

$$zz^* = |z|^2 = 25.$$

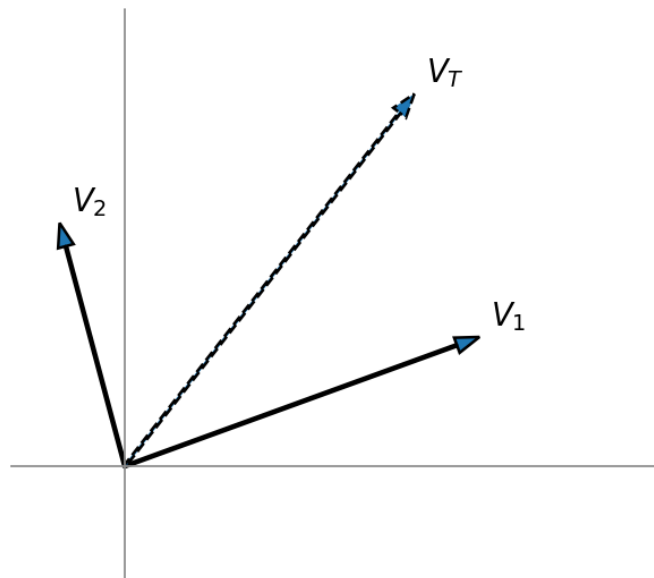
This identity is why conjugation appears in RF power formulas: multiplying a phasor by a conjugate naturally produces real magnitude-squared terms.

Engineering connection. If z were an impedance, $4 + j3 \, \Omega$ would mean 4 ohms of resistance and +3 ohms of reactance.

Worked Example 3 - Add two phasors correctly

Let

$$V_1 = 10\angle 20^\circ \text{ V}, \quad V_2 = 6\angle 105^\circ \text{ V}.$$



Phasor addition

Convert to rectangular form:

$$\begin{aligned} V_1 &= 10(\cos 20^\circ + j\sin 20^\circ) = 9.397 + j3.420, \\ V_2 &= 6(\cos 105^\circ + j\sin 105^\circ) = -1.553 + j5.796. \end{aligned}$$

Add components:

$$V_T = 7.844 + j9.216.$$

Magnitude and phase are

$$\begin{aligned} |V_T| &= \sqrt{7.844^2 + 9.216^2} = 12.10 \text{ V}, \\ \angle V_T &= \tan^{-1}\left(\frac{9.216}{7.844}\right) = 49.6^\circ. \end{aligned}$$

Hence

$$\boxed{V_T \approx 12.10\angle 49.6^\circ \text{ V}.}$$

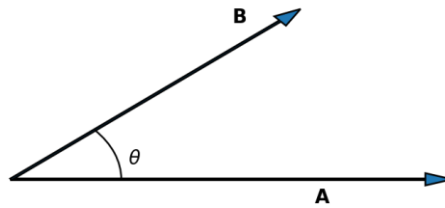
Key lesson. Do not add phasor magnitudes and phases separately. Addition is easiest in rectangular form; multiplication/division is easiest in polar form.

Worked Example 4 - Dot product, projection and work

Let

$$\mathbf{A} = 3\hat{x} + 4\hat{y}, \quad \mathbf{B} = 5\hat{x} + 2\hat{y}.$$

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \quad |\mathbf{A} \times \mathbf{B}| = AB \sin \theta$$



Dot and cross product geometry

The dot product is

$$\mathbf{A} \cdot \mathbf{B} = (3)(5) + (4)(2) = 23.$$

Since

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta,$$

with $A = 5$ and $B = \sqrt{29}$,

$$\cos \theta = \frac{23}{5\sqrt{29}},$$

so

$$\theta \approx 31.5^\circ.$$

The component of $\mathbf{B}$ along $\mathbf{A}$ is

$$B_{\parallel A} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|} = \frac{23}{5} = 4.6.$$

In EM, the same operation appears in work and flux. If a charge moves by $d\mathbf{l}$, the incremental electric work contains $\mathbf{E} \cdot d\mathbf{l}$. The dot product selects the component **along** the path.

Worked Example 5 - Cross product and electromagnetic power direction

Suppose a uniform plane wave has

$$\mathbf{E} = 100\hat{\mathbf{x}} \text{ V/m}, \quad \mathbf{H} = 0.265\hat{\mathbf{y}} \text{ A/m}.$$

The instantaneous Poynting direction is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Using the right-hand rule,

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}},$$

therefore

$$\boxed{\mathbf{S} = 26.5\hat{\mathbf{z}} \text{ W/m}^2}$$

for simultaneous instantaneous values.

The cross product is appropriate because it gives both a magnitude proportional to $EH\sin\theta$ and a direction perpendicular to the two field vectors. For a plane wave, $\mathbf{E} \perp \mathbf{H}$, so $\sin\theta = 1$.

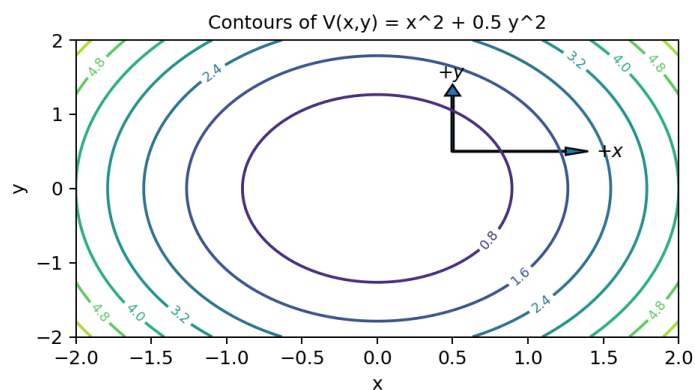
This one calculation links the mathematical foundations directly to later EM power flow.

Common mistake: writing $\mathbf{E} \times \mathbf{B}$ without the required $1/\mu$ factor. The macroscopic Poynting vector is $\mathbf{E} \times \mathbf{H}$.

Worked Example 6 - Interpret a partial derivative of a field

Let a scalar potential be

$$V(x, y) = x^2 + \frac{1}{2}y^2.$$



Scalar-field contours

The partial derivative with respect to x holds y fixed:

$$\frac{\partial V}{\partial x} = 2x.$$

Similarly,

$$\frac{\partial V}{\partial y} = y.$$

The gradient is therefore

$$\nabla V = 2x\hat{x} + y\hat{y}.$$

If this is an electrostatic potential, then

$$\mathbf{E} = -\nabla V = -2x\hat{x} - y\hat{y}.$$

At $(x, y) = (1, 2)$,

$$\boxed{\mathbf{E} = -2\hat{x} - 2\hat{y}}.$$

The contour plot is important: ∇V points normal to contours toward increasing V , whereas $\mathbf{E}$ points in the opposite direction.

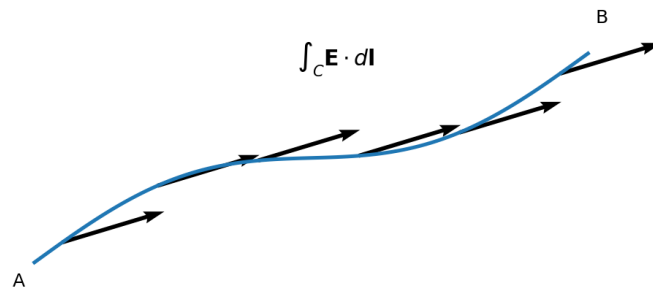
Worked Example 7 - Evaluate a line integral along a simple path

Let

$$\mathbf{E} = 2x\hat{\mathbf{x}} + 3\hat{\mathbf{y}} \text{ V/m,}$$

and move from (0,0) to (2,1) along the straight path

$$\mathbf{r}(t) = 2t\hat{\mathbf{x}} + t\hat{\mathbf{y}}, \quad 0 \leq t \leq 1.$$



Line integral path

Then

$$d\mathbf{l} = (2\hat{\mathbf{x}} + \hat{\mathbf{y}})dt,$$

and along the path $x = 2t$, so

$$\mathbf{E} = 4t\hat{\mathbf{x}} + 3\hat{\mathbf{y}}.$$

Hence

$$\int_C \mathbf{E} \cdot d\mathbf{l} = \int_0^1 (8t + 3) dt = 4 + 3 = 7 \text{ V.}$$

For an electrostatic field, the potential difference is related by

$$V_B - V_A = - \int_A^B \mathbf{E} \cdot d\mathbf{l}.$$

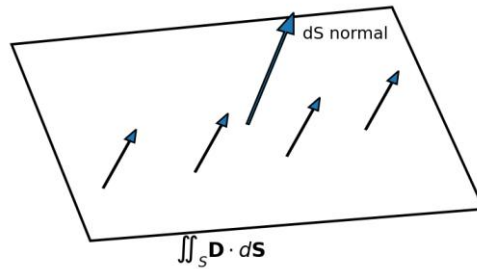
The line integral accumulates only the component of field tangent to the chosen path.

Worked Example 8 - Electric flux through a surface

Suppose a uniform electric flux density is

$$\mathbf{D} = 5\hat{\mathbf{z}} \mu\text{C}/\text{m}^2$$

and passes through a flat area of 0.20 m^2 whose unit normal makes 30° with $+\mathbf{z}$.



Flux through a surface

The flux is

$$\Psi_D = \iint_S \mathbf{D} \cdot d\mathbf{S}.$$

Because the field and normal are uniform,

$$\Psi_D = DA \cos 30^\circ.$$

Therefore

$$\Psi_D = (5 \times 10^{-6})(0.20)(0.866) = 8.66 \times 10^{-7} \text{ C}.$$

So

$$\boxed{\Psi_D = 0.866 \mu\text{C}.$$

The dot product is doing the geometrical work: only the component normal to the surface contributes to flux.

Worked Example 9 - Volume integration of charge density

A cylinder has radius $a = 20$ mm, length $L = 100$ mm, and uniform volume charge density

$$\rho_v = 4 \text{ } \mu\text{C/m}^3.$$

Using cylindrical coordinates,

$$dV = \rho \, d\rho \, d\phi \, dz.$$

The total charge is

$$Q = \iiint_V \rho_v \, dV = \rho_v \int_0^L \int_0^{2\pi} \int_0^a \rho \, d\rho \, d\phi \, dz.$$

Evaluate the integrals:

$$Q = \rho_v \left(\frac{a^2}{2} \right) (2\pi) L = \rho_v \pi a^2 L.$$

Numerically,

$$Q = (4 \times 10^{-6}) \pi (0.020)^2 (0.100) = 5.03 \times 10^{-10} \text{ C}.$$

Thus

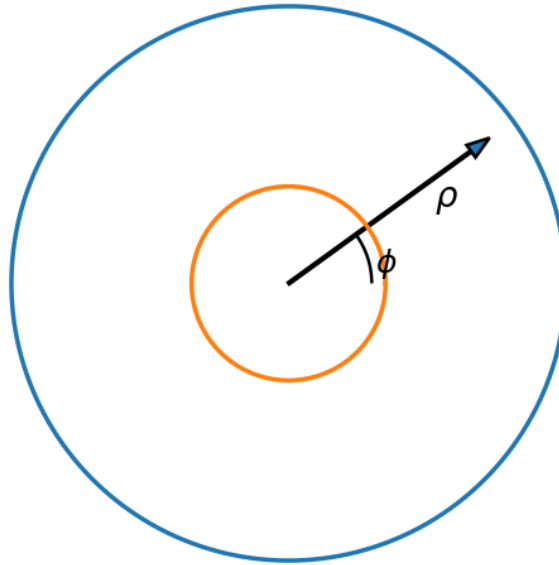
$$\boxed{Q \approx 0.503 \text{ nC.}}$$

The factor ρ in dV is the cylindrical-coordinate Jacobian. Omitting it is a common error.

Worked Example 10 - Choose coordinates that match the geometry

Consider an ideal coaxial transmission line with inner radius a , outer-conductor inner radius b , and cylindrical symmetry.

Coaxial geometry



Cylindrical coordinates for coax

Cartesian coordinates would force us to describe a circular geometry using x and y . Cylindrical coordinates align directly with the physics. By symmetry the electrostatic potential depends only on ρ :

$$V = V(\rho).$$

The radial electric field is

$$\mathbf{E} = -\frac{dV}{d\rho} \hat{\rho}.$$

For the familiar coax solution,

$$V(\rho) = V_0 \frac{\ln(b/\rho)}{\ln(b/a)},$$

so

$$\boxed{\mathbf{E}(\rho) = \frac{V_0}{\rho \ln(b/a)} \hat{\rho}.$$

Notice how the coordinate choice exposes the physics: no ϕ or z dependence remains. This is the deeper reason coordinate systems matter in EM - a good coordinate system can turn a three-variable problem into a one-variable problem.

Foundations checkpoint. You should now be able to move fluently between vector components, complex forms, dot/cross products, partial derivatives and line/surface/volume integrals, while keeping the physical meaning of each operation visible.