

Introduction to Computational Electromagnetics

Electromagnetic Theory Course — Interfaces, Transmission Lines, Waveguides and Numerical EM — Document 14 of 23

1. Why computational electromagnetics is needed

Maxwell's equations are compact, but exact closed-form solutions exist only for geometries with enough symmetry. Real antennas, radomes, PCB transitions, connectors, scattering bodies and dielectric structures quickly become analytically intractable.

Computational electromagnetics (CEM) replaces the continuous field problem by a finite algebraic problem that a computer can solve.

The starting equations may be differential, for example

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = -j\omega \mathbf{J},$$

or integral, for example a boundary equation relating current to the field through a Green's function.

The solver then follows a broad sequence:

geometry and materials → **discretization** → **basis/unknowns** → **matrix or time-step equations** → **numerical solution** → **field/power/S-parameter post-processing**.

The central intellectual step is discretization: replacing infinitely many possible field values by a finite set of unknown coefficients.

This introduces approximation, so every numerical result has two questions attached:

1. Did we solve the discrete equations accurately?
2. Do the discrete equations approximate the original Maxwell problem accurately enough?

Those are different questions. A linear solver can converge perfectly to the wrong physical answer if the mesh, basis, boundary conditions or formulation are inadequate.

Core mental model: a CEM solver is a controlled translation from Maxwell's continuous operators into finite algebra, together with a disciplined process for estimating the error introduced by that translation.

2. Discretization, meshes, basis functions and degrees of freedom

A continuous field such as $\mathbf{E}(\mathbf{r})$ has infinitely many possible values. Numerical methods approximate it using a finite set of basis functions:

$$\mathbf{E}(\mathbf{r}) \approx \sum_{n=1}^N e_n \mathbf{f}_n(\mathbf{r}).$$

The coefficients e_n become the numerical unknowns or **degrees of freedom (DoF)**.

Different methods choose different basis structures. FEM often uses basis functions associated with mesh edges or nodes. MoM uses surface or wire-current basis functions. FDTD stores field components on a spatial grid and advances them directly in time.

A mesh divides the geometry into cells: triangles/tetrahedra, quadrilaterals/hexahedra, or structured Yee cells. Mesh density must resolve both geometry and field variation.

A common rule of thumb is several elements per wavelength, but that is not a universal accuracy criterion. High material contrast, sharp corners, small gaps, skin-depth effects and near-singular fields can require much finer local resolution.

The correct mathematical space of the basis functions also matters. For full-vector Maxwell FEM, edge elements are commonly used because they represent tangential field continuity correctly and suppress many nonphysical spurious modes that appear with naive nodal vector interpolation.

Discretization therefore contains physics. It is not just “draw a finer grid.” The mesh topology, basis functions and continuity properties must match the operators and boundary conditions of Maxwell’s equations.

3. Linear systems: from fields to $\mathbf{Ax} = \mathbf{b}$

Many driven frequency-domain EM formulations become a linear algebra problem

$$\mathbf{Ax} = \mathbf{b}.$$

The vector $\mathbf{x}$ contains unknown field or current coefficients. The source/excitation is represented by $\mathbf{b}$. The matrix $\mathbf{A}$ contains geometry, material properties, differential or integral operators, and boundary conditions.

For MoM one often writes

$$\mathbf{ZI} = \mathbf{V},$$

where $\mathbf{Z}$ is a dense interaction matrix.

For FEM a driven system may look schematically like

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{e} = \mathbf{f}.$$

The matrix can be enormous. Numerical algorithms therefore solve the system without explicitly forming

$$\mathbf{A}^{-1}.$$

Direct methods such as LU factorization are robust but can use large memory. Iterative methods such as GMRES or conjugate-gradient variants can handle much larger systems when the matrix properties and preconditioning are favourable.

Conditioning matters. A matrix with a large condition number can amplify numerical error and make iterative convergence slow. Geometry, frequency, mesh quality and formulation all influence conditioning.

Sparse versus dense structure is also fundamental. Local differential methods such as FEM often produce sparse matrices because each basis function interacts directly with nearby neighbours. Integral-equation MoM matrices are usually dense because every source element communicates with every observation element through the Green's function.

4. Eigenvalue problems: finding natural modes

A driven simulation asks: “what field results from this source at this frequency?” An eigenmode simulation asks: “what field patterns can the structure support naturally?”

The discrete form commonly becomes

$$\mathbf{Ax} = \lambda \mathbf{Bx}.$$

The vector $\mathbf{x}$ is an eigenvector: a mode shape. The scalar λ is the associated eigenvalue.

For a cavity FEM problem, a simplified form is

$$\mathbf{Ke} = \omega^2 \mathbf{Me}.$$

Thus each eigenvalue gives a resonant ω^2 and each eigenvector gives the sampled electric-field pattern.

For a waveguide cross-section, the eigenvalue may instead correspond to cutoff wavenumber or propagation constant. This is the numerical generalization of the analytical TE/TM eigenfunctions of rectangular waveguide.

Eigenproblems also appear in characteristic-mode analysis of antennas, photonic crystals, periodic structures and multi-conductor transmission lines.

Numerically, one rarely wants every eigenpair of a huge matrix. Algorithms target modes near a requested frequency or the smallest/largest eigenvalues of interest.

A critical verification step is checking that a computed mode satisfies the boundary conditions and has the expected symmetry, field continuity and energy behaviour. Numerical eigenproblems can produce spurious solutions if the formulation or basis space is inappropriate.

5. Method of Moments: integral equations and dense coupling

The Method of Moments (MoM) is especially natural for radiation and scattering problems because it often reduces a 3D exterior field problem to unknown currents on surfaces or wires.

A generic integral equation may be written

$$\mathcal{L}\{J\} = V.$$

Approximate the unknown current by basis functions:

$$J \approx \sum_{n=1}^N I_n f_n$$

Then enforce the equation using testing functions w_m . This produces

$$\boxed{\mathbf{Z}\mathbf{I} = \mathbf{V}}$$

with

$$Z_{mn} = \langle w_m, \mathcal{L}\{f_n\} \rangle.$$

The electromagnetic interaction is carried by a Green's function. In a homogeneous 2D Helmholtz problem, for example,

$$G(\rho) = \frac{j}{4} H_0^{(1)}(k\rho)$$

under one common time convention.

Because every basis function radiates to every testing location, $\mathbf{Z}$ is generally dense. Self terms require special care because the Green's function is singular as source and observation positions coincide.

MoM is powerful for PEC antennas and scattering objects because only surfaces need to be discretized. The cost is dense-matrix storage/solution and the need for suitable Green's functions for the surrounding medium.

Document 16 will develop this method in detail for the 2D EM case.

6. Finite Element Method: local weak-form physics

FEM begins with a differential equation and converts it into a **weak form** by multiplying by a test function and integrating over the domain. Integration by parts lowers derivative order and exposes natural boundary terms.

A common frequency-domain electric-field equation is

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = -j\omega \mathbf{J}.$$

After expansion in basis functions and testing, one obtains a sparse system such as

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{e} = \mathbf{f}.$$

FEM handles complicated geometry and inhomogeneous materials extremely well because material parameters can vary element by element.

For vector electromagnetics, **edge elements** are important because the tangential continuity of $\mathbf{E}$ across element boundaries matches Maxwell's physics.

Open-region radiation problems require truncating the computational domain. This is usually done with radiation conditions or a **perfectly matched layer (PML)** that absorbs outgoing waves with minimal reflection.

FEM meshes can be locally refined around small details without refining the entire domain. Higher-order basis functions can also increase accuracy per element.

Its disadvantages include meshing the entire volume, sometimes very large sparse systems, and sensitivity to mesh quality. Nevertheless, FEM is one of the most versatile general-purpose full-wave methods.

7. Finite-Difference Time-Domain: stepping Maxwell through time

FDTD discretizes Maxwell's curl equations directly in space and time. In the classic Yee scheme, electric and magnetic field components are staggered spatially and temporally.

Conceptually,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

updates $\mathbf{H}$ from known $\mathbf{E}$, then

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

updates $\mathbf{E}$ from the new $\mathbf{H}$.

The result is an explicit leapfrog time-stepping algorithm.

A major advantage is broadband behaviour: one short pulse can excite a wide frequency range, and Fourier transformation of the time signals yields frequency-domain response.

The time step must satisfy a stability limit. For a simple 3D Cartesian grid, the Courant condition is approximately

$$\Delta t \leq \frac{1}{v \sqrt{1/\Delta x^2 + 1/\Delta y^2 + 1/\Delta z^2}}.$$

Fine spatial resolution therefore forces small time steps.

FDTD also exhibits **numerical dispersion**: the discrete wave velocity depends slightly on direction and grid resolution. Curved boundaries can be staircase approximations unless conformal techniques are used.

PML is commonly used to terminate the grid. FDTD is excellent for broadband transients, pulse propagation and complex nonlinear/time-varying problems, but electrically large high-Q problems can require very many time steps.

8. Accuracy: convergence, dispersion, singularities and conditioning

A computed field is never trustworthy merely because the solver produced a colourful plot. Accuracy must be demonstrated.

Convergence means the result approaches a stable value as numerical resolution improves. Typical studies refine the mesh, increase basis order, reduce time step, enlarge the computational domain or increase the number of modes.

For a quantity Q , one may compare successive refinements:

$$\epsilon_r = \frac{|Q_{fine} - Q_{coarse}|}{|Q_{fine}|}.$$

A small change is evidence of convergence, though not proof of correctness.

Numerical dispersion means the computed phase constant differs from the physical one. It is especially important in FDTD and low-order FEM over long propagation distances.

Singular fields occur at sharp PEC edges and at coincident source/observation points in integral equations. Naive quadrature or sampling can fail badly there.

Conditioning controls how errors in data and floating-point arithmetic affect the solution. A badly conditioned matrix may need scaling, reformulation or preconditioning.

Boundary truncation introduces another error source. A PML that is too thin or poorly parameterized reflects energy back into the domain.

Mesh quality, material representation, port placement and interpolation can all dominate the final error even when the matrix solver residual is tiny.

Engineering habit: always separate solver convergence from physical validation. A residual of 10^{-10} only tells you that the discrete equations were solved accurately.

9. Verification and validation: how to trust a solver

Verification asks whether the equations were implemented and solved correctly. **Validation** asks whether the model represents the real physical system well enough for the intended use.

Verification should begin with problems having known answers. Examples include

- plane-wave propagation in a uniform medium;
- reflection from a PEC plane;
- coaxial characteristic impedance;
- rectangular-waveguide TE_{10} cutoff;
- resonant frequency of a simple cavity;
- scattering from canonical cylinders or spheres.

Cross-check independent identities. In a lossless rectangular guide, for example,

$$\beta^2 + k_c^2 = k^2$$

and

$$v_p v_g = v^2.$$

Power should also balance. For a passive lossless two-port,

$$|S_{11}|^2 + |S_{21}|^2 \approx 1$$

when only one propagating mode exists at each port.

Validation compares against measurement or a trusted experimental reference. Here uncertainties in material properties, connector launches, manufacturing tolerances and calibration matter.

A disciplined workflow records geometry, material data, mesh settings, boundary conditions, solver tolerances and convergence evidence. Reproducibility is part of numerical credibility.

When two solvers disagree, do not immediately assume one is wrong. They may be solving slightly different physical models, using different reference planes, port definitions, material conventions or normalization.

10. Choosing a method and degree-level checkpoint

No single CEM technique is best for every problem.

MoM is attractive for open-region PEC radiation/scattering when surfaces are much smaller than the surrounding free-space volume. It produces dense matrices.

FEM is strong for complicated bounded geometries, dielectric loading, waveguides, resonators and frequency-domain multi-material problems. It produces sparse matrices.

FDTD is strong for broadband transient behaviour and direct time-domain visualization, at the cost of a volumetric grid and stability-limited time step.

Hybrid solvers combine methods when one part of a problem suits one formulation and another part suits something else.

Common mistakes

- Treating mesh density as the only accuracy control.
- Assuming a converged linear-solver residual proves physical correctness.
- Explicitly computing $\mathbf{A}^{-1}$ instead of solving $\mathbf{Ax} = \mathbf{b}$.
- Ignoring port normalization or reference-plane definitions.
- Comparing different solvers without matching geometry/material conventions.
- Failing to perform a mesh or timestep convergence study.

Degree-level checkpoint

1. Explain what discretization means physically and mathematically.
2. Why are MoM matrices usually dense while FEM matrices are usually sparse?
3. Distinguish a driven linear system from an eigenvalue problem.
4. Explain why edge elements are useful in vector FEM.
5. State the main advantage and main limitation of FDTD.
6. Define verification and validation and give one EM example of each.
7. For a PEC antenna in free space and a dielectric-filled cavity, choose plausible numerical methods and justify each choice.

Teach-back target: you should be able to explain how Maxwell's continuous field equations become finite algebra, compare MoM/FEM/FDTD intelligently, and describe the evidence required before trusting a numerical EM result.