

Plane Waves, Polarization, Impedance and Power

Electromagnetic Theory Course — Maxwell and Wave Theory — Document 10 of 23

1. What a uniform plane wave really means

A **uniform plane wave** is the simplest travelling electromagnetic wave. It is an idealisation, but an extremely useful one because many real fields become approximately plane over a sufficiently small region far from an antenna.

Suppose a wave travels in the $+z$ direction. A uniform plane wave has no variation across any plane of constant z :

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0.$$

Its field changes only with z and time. A simple linearly polarised example is

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}E_0 \cos(\omega t - \beta z),$$

with associated magnetic field

$$\mathbf{H}(z, t) = \hat{\mathbf{y}}H_0 \cos(\omega t - \beta z).$$

The two fields are transverse to the propagation direction:

$$\mathbf{E} \perp \mathbf{H}, \quad \mathbf{E} \perp \hat{\mathbf{z}}, \quad \mathbf{H} \perp \hat{\mathbf{z}}.$$

The three directions form a right-handed set. For this example,

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}},$$

so the wave transports energy toward $+z$.

In phasor form, using the course convention $e^{j\omega t}$,

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}}E_0 e^{-j\beta z}, \\ \tilde{\mathbf{H}}(z) &= \hat{\mathbf{y}}H_0 e^{-j\beta z}.\end{aligned}$$

The exponential $e^{-j\beta z}$ contains the spatial phase variation. At a fixed position, the suppressed factor $e^{j\omega t}$ provides the time oscillation.

Why “plane”?

A surface of constant phase satisfies

$$\omega t - \beta z = \text{constant}.$$

At a fixed time, this is a plane $z = \text{constant}$. Every point on that plane has the same phase. The planes move through space as time advances.

A spherical wave from a small antenna becomes locally almost planar when the observation distance is large compared with the dimensions of the region being considered. This is why plane-wave ideas are central to antenna measurements, propagation, radar, EMC and link analysis.

Core mental model: a uniform plane wave is a travelling field pattern whose phase fronts are planes and whose $\mathbf{E}$ and $\mathbf{H}$ fields lie entirely transverse to the direction of propagation.

2. Deriving the E-H relationship from Maxwell's equations

Take a lossless plane wave travelling in $+z$:

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}E_0 e^{-j\beta z}.$$

With the $e^{j\omega t}$ convention, Faraday's law is

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}.$$

Because E_x varies only with z , the curl has only a y component:

$$\nabla \times \tilde{\mathbf{E}} = \hat{\mathbf{y}} \frac{\partial E_x}{\partial z} = -j\beta \hat{\mathbf{y}} E_0 e^{-j\beta z}.$$

Therefore

$$-j\beta \hat{\mathbf{y}} E_0 e^{-j\beta z} = -j\omega\mu\tilde{\mathbf{H}}.$$

Canceling the common factor gives

$$\tilde{\mathbf{H}} = \hat{\mathbf{y}} \frac{\beta}{\omega\mu} E_0 e^{-j\beta z}.$$

For a lossless medium,

$$\beta = \omega\sqrt{\mu\epsilon},$$

so

$$\frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta.$$

Hence

$$\boxed{\tilde{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{k}} \times \tilde{\mathbf{E}}}$$

and equivalently

$$\boxed{\tilde{\mathbf{E}} = \eta \tilde{\mathbf{H}} \times \hat{\mathbf{k}}}.$$

These are not empirical rules. They are consequences of Maxwell's curl equations.

For the example,

$$\hat{\mathbf{k}} = \hat{\mathbf{z}},$$

and

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}},$$

which gives the correct magnetic-field direction.

Direction test

A useful check is

$$\boxed{\mathbf{E} \times \mathbf{H} \text{ points in the direction of propagation.}}$$

If your calculated $\mathbf{E}$, $\mathbf{H}$ and propagation direction do not form that right-handed relationship for a forward lossless plane wave, revisit the sign convention or cross product.

A deeper point

Once the direction of propagation and one transverse field are known, the other field is not independent. Maxwell's equations determine it. A plane wave therefore has fewer independent degrees of freedom than six arbitrary field components might suggest.

3. Intrinsic impedance: what the ratio E/H means

For a lossless homogeneous medium, the **intrinsic impedance** is

$$\eta = \sqrt{\frac{\mu}{\epsilon}}.$$

For a travelling plane wave,

$$\frac{|\vec{E}|}{|\vec{H}|} = \eta.$$

The units are ohms:

$$\frac{\text{V/m}}{\text{A/m}} = \Omega.$$

This resemblance to circuit resistance is useful, but intrinsic impedance is not a resistor and it is not the impedance “between two points”. It is a **local ratio between electric and magnetic field amplitudes in a travelling wave**.

In vacuum,

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.73 \, \Omega.$$

For a nonmagnetic lossless dielectric,

$$\mu_r \approx 1,$$

so

$$\eta \approx \frac{\eta_0}{\sqrt{\epsilon_r}}.$$

For example, if $\epsilon_r = 4$,

$$\eta \approx \frac{376.73}{2} \approx 188.4 \, \Omega.$$

The electric field required for a given magnetic field is therefore smaller than in free space.

Lossy medium

When conductivity is significant,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

Now η is complex. Its magnitude gives the field-amplitude ratio, while its phase describes the phase offset between **E** and **H**.

For a good conductor,

$$\sigma \gg \omega\epsilon,$$

and approximately

$$\eta \approx (1 + j) \sqrt{\frac{\omega\mu}{2\sigma}}.$$

Thus **E** and **H** are no longer exactly in phase.

Intrinsic impedance versus transmission-line impedance

Do not automatically identify η with a cable or waveguide characteristic impedance Z_0 . They are related ideas, but not identical quantities. η belongs to a propagating field in a material. A transmission-line Z_0 is a voltage-to-current ratio for a particular guided mode and geometry.

In free-space plane-wave work, however, the ratio

$$E/H \approx 377 \, \Omega$$

is so fundamental that it often serves as a quick consistency check on calculated far fields.

Teach-backtest: explain why “377 ohms of free space” does not mean that space behaves like a 377-ohm lumped resistor.

4. Phase, wavelength, wave vector and direction of travel

A three-dimensional plane-wave phasor is commonly written

$$\tilde{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}},$$

where the **wave vector** is

$$\boxed{\mathbf{k} = \beta \hat{\mathbf{k}}}.$$

Its magnitude is the phase constant,

$$|\mathbf{k}| = \beta,$$

and its direction is the direction of phase propagation.

Restoring time dependence gives

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}\}.$$

Constant phase requires

$$\omega t - \mathbf{k} \cdot \mathbf{r} = \text{constant}.$$

The wavelength is

$$\boxed{\lambda = \frac{2\pi}{\beta}}.$$

The phase velocity is

$$\boxed{v_p = \frac{\omega}{\beta} = f\lambda}.$$

Example: propagation in an arbitrary direction

Suppose

$$\hat{\mathbf{k}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{z}}).$$

Then

$$\mathbf{k} \cdot \mathbf{r} = \frac{\beta}{\sqrt{2}}(x + z).$$

Surfaces of constant phase satisfy

$$x + z = \text{constant}.$$

These are planes perpendicular to $\hat{\mathbf{k}}$.

The electric field must also be transverse:

$$\mathbf{k} \cdot \mathbf{E}_0 = 0.$$

Therefore an arbitrary vector cannot be chosen as the polarization vector; it must lie in the plane perpendicular to propagation.

Spatial phase is not attenuation

In a lossless wave,

$$e^{-j\beta z}$$

has magnitude one. It changes phase but not amplitude. Attenuation requires a real exponential factor such as

$$e^{-\alpha z}.$$

A common reading error is to see a negative exponent and assume “decay”. The factor j changes everything: $e^{-j\beta z}$ rotates in the complex plane; $e^{-\alpha z}$ shrinks in magnitude.

Sign convention reminder

With $e^{j\omega t}$ suppressed, $e^{-j\beta z}$ represents propagation toward +z. If a textbook uses $e^{-j\omega t}$ instead, the spatial signs are normally reversed. Always identify the time convention before interpreting a propagation factor.

5. Linear polarization and polarization vectors

Polarization describes the path traced by the tip of the electric-field vector at a fixed point in space as time progresses.

For a wave travelling in $+z$, write the transverse phasor as

$$\tilde{\mathbf{E}} = \hat{\mathbf{x}}\tilde{E}_x + \hat{\mathbf{y}}\tilde{E}_y.$$

If the two components are in phase or exactly 180° out of phase, their ratio is real and the field remains along a fixed line. The wave is **linearly polarised**.

For example,

$$\tilde{\mathbf{E}} = E_0 (\hat{\mathbf{x}}\cos\psi + \hat{\mathbf{y}}\sin\psi)$$

is linearly polarised at angle ψ to the x axis.

Its corresponding magnetic field is

$$\tilde{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{z}} \times \tilde{\mathbf{E}}.$$

Using the cross products

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}},$$

we obtain

$$\tilde{\mathbf{H}} = \frac{E_0}{\eta} (\hat{\mathbf{y}}\cos\psi - \hat{\mathbf{x}}\sin\psi).$$

Thus rotating the electric polarization rotates the magnetic field with it while preserving the orthogonal **E-H-k** geometry.

Polarization mismatch

A linearly polarised receiving antenna responds most strongly when its polarization is aligned with the incident electric field. If the receive polarization is rotated by angle ψ from the incident linear polarization, the received field projection is proportional to

$$\cos\psi.$$

Power is proportional to the square of field amplitude, so the polarization coupling factor is

$$\boxed{\text{PLF} = \cos^2\psi}.$$

Examples:

- $\psi = 0^\circ$: full coupling;
- $\psi = 45^\circ$: half the power, a 3 dB loss;
- $\psi = 90^\circ$: ideal orthogonal linear polarizations have zero coupling.

Real antennas have finite cross-polar response, so actual isolation is never infinite.

Why this matters in RF

Polarization is not decorative geometry. It directly affects received power, antenna orientation, satellite links, radar scattering, propagation measurements and MIMO behaviour. The field vector must therefore be treated as a vector, not merely as a scalar signal level.

6. Circular and elliptical polarization

If the orthogonal components of the electric field have a relative phase, the direction of $\mathbf{E}$ can rotate with time.

Let

$$\tilde{\mathbf{E}} = \hat{x}A + \hat{y}B e^{j\delta}.$$

The time-domain components are

$$\begin{aligned} E_x(t) &= A \cos \omega t, \\ E_y(t) &= B \cos(\omega t + \delta). \end{aligned}$$

Three important cases follow.

Linear polarization

If

$$\delta = 0 \quad \text{or} \quad \pi,$$

the components reach maxima and zero crossings together. Their ratio remains fixed, so the field traces a line.

Circular polarization

If

$$A = B$$

and

$$\delta = \pm 90^\circ,$$

then the magnitude of the total electric field is constant while its direction rotates. The tip of $\mathbf{E}$ traces a circle.

For example,

$$\tilde{\mathbf{E}} = E_0(\hat{x} + j\hat{y})$$

contains equal orthogonal components in quadrature.

Whether this is called right-hand or left-hand circular polarization depends on the adopted viewing and time convention. **Always state the convention** rather than relying on an unlabeled handedness name.

Elliptical polarization

The general case is elliptical. This occurs when

- the orthogonal amplitudes are unequal;
- or the relative phase is neither 0, 180°, nor the equal-amplitude quadrature case;
- or both.

Linear and circular polarization are therefore special cases of elliptical polarization.

Axial ratio

For an ellipse, the **axial ratio** is

$$\text{AR} = \frac{\text{major-axis field amplitude}}{\text{minor-axis field amplitude}} \geq 1.$$

Circular polarization has

$$\text{AR} = 1,$$

while ideal linear polarization corresponds to an infinite axial ratio. In antenna engineering axial ratio is often expressed in decibels:

$$\text{AR}_{\text{dB}} = 20 \log_{10}(\text{AR}).$$

Why phasors make polarization easier

Without phasors, each component must be tracked as a time-dependent sinusoid. With phasors, the relative amplitude and phase are encoded directly in the complex vector components. Polarization is therefore one of the clearest examples of complex vector notation carrying genuine physical information.

7. The Poynting vector: electromagnetic power flow

The instantaneous electromagnetic power-flow density is the **Poynting vector**:

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t)$$

with units

$$\frac{\text{V}}{\text{m}} \frac{\text{A}}{\text{m}} = \frac{\text{W}}{\text{m}^2}.$$

For the simple plane wave

$$\mathbf{E} = \hat{\mathbf{x}}E, \quad \mathbf{H} = \hat{\mathbf{y}}H,$$

we obtain

$$\mathbf{S} = \hat{\mathbf{z}}EH.$$

So the cross product supplies both the **amount of power per unit area** and the **direction in which energy flows**.

For a lossless plane wave,

$$H = \frac{E}{\eta},$$

so instantaneously

$$S = \frac{E^2}{\eta}.$$

Because E is sinusoidal, this instantaneous power density also oscillates during the RF cycle. In engineering we usually want its time average.

Time-average power using peak phasors

With peak-amplitude phasors,

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \tilde{\mathbf{E}} \times \text{conj}(\tilde{\mathbf{H}}) \}.$$

Here $\text{conj}(\tilde{\mathbf{H}})$ means the **complex conjugate** of the magnetic-field phasor; many EM texts denote the same operation with a superscript asterisk.

For a lossless plane wave with peak field magnitude E_0 ,

$$\langle S \rangle = \frac{E_0^2}{2\eta} = \frac{\eta H_0^2}{2}.$$

If RMS phasors are used instead, the factor $1/2$ disappears:

$$\langle S \rangle = \frac{E_{\text{rms}}^2}{\eta}.$$

This is why a document or software package must state whether phasor magnitudes are peak or RMS.

Total power through a surface

Power density becomes total power by integrating the normal component over an area:

$$P = \iint_A \langle \mathbf{S} \rangle \cdot d\mathbf{A}.$$

This equation appears everywhere in EM: waveguide power, antenna radiation, aperture power, absorber calculations and solver post-processing.

Core mental model: voltage-times-current is the circuit view of power; $\mathbf{E} \times \mathbf{H}$ is the field view.

8. Complex Poynting vector, stored energy and lossy media

For sinusoidal fields it is useful to retain the full complex quantity

$$\mathbf{S}_c = \frac{1}{2} \tilde{\mathbf{E}} \times \text{conj}(\tilde{\mathbf{H}})$$

when peak phasors are used.

Its real part is the average transported power density:

$$\text{Re}\{\mathbf{S}_c\} = \langle \mathbf{S} \rangle.$$

Its imaginary part is associated with **reactive energy exchange** rather than net time-average transport. Near antennas, discontinuities and resonant structures, electric and magnetic energy can slosh back and forth strongly even when the net outward real power is much smaller.

Energy density in a lossless nondispersive medium

The instantaneous electric and magnetic energy densities are

$$u_e = \frac{1}{2} \epsilon E^2,$$

and

$$u_m = \frac{1}{2} \mu H^2.$$

For a lossless plane wave,

$$E = \eta H, \quad \eta^2 = \frac{\mu}{\epsilon},$$

so

$$\epsilon E^2 = \mu H^2.$$

Therefore the electric and magnetic energy densities are equal for the ideal travelling plane wave.

Using peak phasors, their time averages are

$$\begin{aligned} \langle u_e \rangle &= \frac{1}{4} \epsilon |\tilde{E}|^2, \\ \langle u_m \rangle &= \frac{1}{4} \mu |\tilde{H}|^2. \end{aligned}$$

What changes in a lossy medium?

If

$$\gamma = \alpha + j\beta,$$

then the field amplitude contains

$$e^{-\alpha z}.$$

Power density is proportional to field amplitude squared, so for a simple uniform wave its transported power decays approximately as

$$e^{-2\alpha z}.$$

The missing electromagnetic power is not destroyed mathematically; it is converted into other forms, commonly heat through conduction loss.

For conduction current,

$$\mathbf{J} = \sigma \mathbf{E},$$

and the local time-average dissipated power density for peak phasors is

$$\langle p_{\text{loss}} \rangle = \frac{1}{2} \sigma |\tilde{\mathbf{E}}|^2.$$

This connects wave attenuation directly to material heating and is important in RF loss, dielectric heating, shielding and skin-effect calculations.

9. Worked RF examples: from field strength to power density

The plane-wave relationships let us move directly between electric field, magnetic field and power density.

Example 1: a 10 V/m free-space field

Take an RMS electric field

$$E_{\text{rms}} = 10 \text{ V/m.}$$

In free space,

$$\eta_0 \approx 376.73 \Omega,$$

so

$$H_{\text{rms}} = \frac{E_{\text{rms}}}{\eta_0} \approx 26.5 \text{ mA/m.}$$

The average power density is

$$\langle S \rangle = \frac{E_{\text{rms}}^2}{\eta_0} = \frac{100}{376.73} \approx 0.265 \text{ W/m}^2.$$

The same answer follows from

$$\langle S \rangle = E_{\text{rms}} H_{\text{rms}}.$$

Example 2: field strength from known power density

If a far-field region has

$$\langle S \rangle = 1 \text{ mW/m}^2,$$

then

$$E_{\text{rms}} = \sqrt{\eta_0 \langle S \rangle} \approx 0.614 \text{ V/m.}$$

This is the basis of the familiar far-field conversion between power density and electric field strength.

Example 3: polarization mismatch

Suppose the available received power with perfect alignment would be

$$P_0 = 100 \mu\text{W.}$$

If the receive antenna is rotated by 60° relative to the incident linear polarization,

$$\text{PLF} = \cos^2 60^\circ = 0.25.$$

Hence

$$P_r = 25 \mu\text{W.}$$

The polarization mismatch loss is

$$L_p = -10 \log_{10}(0.25) \approx 6.02 \text{ dB.}$$

Far-field link connection

For an isotropic radiator with transmitted power P_t , the ideal free-space power density at range r is

$$\langle S \rangle = \frac{P_t}{4\pi r^2}.$$

An antenna with gain G_t in the observation direction gives

$$\langle S \rangle = \frac{P_t G_t}{4\pi r^2}.$$

A receiving antenna then extracts power according to its effective aperture and polarization coupling. Thus ordinary RF link -budget quantities are directly connected to the underlying plane-wave field theory.

10. Synthesis and degree-level checkpoint

For a uniform plane wave in a homogeneous lossless medium, the central relationships are

$$\tilde{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{k}} \times \tilde{\mathbf{E}},$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}},$$

$$\beta = \omega \sqrt{\mu \epsilon},$$

and

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re} \{ \tilde{\mathbf{E}} \times \text{conj}(\tilde{\mathbf{H}}) \}$$

for peak phasors.

The geometry is equally important:

$$\mathbf{E} \perp \mathbf{H}, \quad \mathbf{E} \perp \hat{\mathbf{k}}, \quad \mathbf{H} \perp \hat{\mathbf{k}},$$

and

$$\mathbf{E} \times \mathbf{H}$$

points in the direction of power flow for the simple forward lossless wave.

Polarization is encoded in the relative amplitudes and phases of the transverse electric-field components. Linear, circular and elliptical polarization are therefore not separate kinds of electromagnetic law; they are different trajectories of the same vector field.

Degree-level checkpoint

You should now be able to explain or derive the following:

1. Define a uniform plane wave and explain why its constant-phase surfaces are planes.
2. Starting from Faraday's law, derive the E/H relationship for a +z travelling plane wave.
3. Explain physically what intrinsic impedance means and why it has units of ohms.
4. State the approximate free-space intrinsic impedance and use it to convert between E and H .
5. Interpret $\mathbf{k} \cdot \mathbf{r}$ geometrically for a wave travelling in an arbitrary direction.
6. Distinguish a phase factor $e^{-j\beta z}$ from an attenuation factor $e^{-\alpha z}$.
7. Determine whether two orthogonal field components produce linear, circular or elliptical polarization.
8. Derive the $\cos^2 \psi$ polarization mismatch factor for two linearly polarised antennas.
9. Explain why the Poynting vector is a cross product and state its units.
10. Convert between peak and RMS forms of the average plane-wave power equation.
11. Explain the meaning of the real and imaginary parts of the complex Poynting vector.
12. Calculate free-space power density from a known electric field and calculate electric field from a known power density.

Teaching synthesis

A useful way to teach this material is to keep four questions visible throughout:

Where is the wave going? — $\hat{\mathbf{k}}$.

Which way is the electric field pointing? — polarization.

What magnetic field must accompany it? — Maxwell plus η .

Where is the energy going, and how much? — $\mathbf{E} \times \mathbf{H}$.

If those four questions remain clear, most plane-wave formulas become consequences rather than isolated facts to memorise.

Core Electromagnetics complete: Documents 5–10 built the path from field quantities through Maxwell's equations to propagating waves, polarization, impedance and power. The next section applies this foundation to interfaces, transmission lines, waveguides and numerical EM.