

From Maxwell's Equations to the Electromagnetic Wave Equation

Electromagnetic Theory Course — Maxwell and Wave Theory — Document 9 of 23

1. The central idea: Maxwell's equations contain waves

An electromagnetic wave is not an extra assumption added to Maxwell's equations. **Wave propagation is already contained inside them.** Once electric and magnetic fields are allowed to vary with time, Faraday's law says that a changing magnetic field produces a circulating electric field, while the Ampère-Maxwell law says that a changing electric field produces a circulating magnetic field. The two fields can therefore sustain one another as a disturbance moves through space.

In a simple homogeneous, linear, source-free medium we use

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H},$$

and set free charge and conduction current to zero:

$$\rho_v = 0, \quad \mathbf{J} = 0.$$

The relevant time-domain Maxwell equations are then

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t}, \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t},\end{aligned}$$

with the divergence conditions

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0$$

when ϵ and μ are spatially uniform.

The key mathematical operation is to take the **curl of a curl**. This introduces the second spatial derivatives needed for a wave equation. The vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

then converts that curl-of-curl expression into a vector Laplacian. In a source-free homogeneous region, the divergence term vanishes, leaving a direct relationship between second derivatives in space and second derivatives in time.

The result will have the same mathematical structure as the ordinary one-dimensional wave equation

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0,$$

but now ψ is replaced by the vector fields $\mathbf{E}$ and $\mathbf{H}$.

Assumptions matter

The clean derivation in the next pages initially assumes a **linear, homogeneous, isotropic, source-free, lossless medium**. Later we relax the lossless assumption and introduce conductivity. At material interfaces, near sources, or in spatially varying media, extra terms can appear and boundary conditions become essential.

Core mental model: Maxwell's curl equations couple $\mathbf{E}$ and $\mathbf{H}$. Taking another curl eliminates one field and reveals that the remaining field satisfies a wave equation.

2. Deriving the time-domain wave equation for the electric field

Start with Faraday's law in a homogeneous medium:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}.$$

Take the curl of both sides:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}).$$

Now substitute the source-free Ampère-Maxwell law,

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t},$$

to obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Use the vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

Gauss's law in a charge-free homogeneous region gives

$$\nabla \cdot \mathbf{D} = 0.$$

Since $\mathbf{D} = \epsilon \mathbf{E}$ and ϵ is constant,

$$\nabla \cdot \mathbf{E} = 0.$$

Therefore

$$-\nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

and hence

$$\boxed{\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

This is the **electromagnetic wave equation for the electric field** in the assumed medium.

Compare it with the generic wave equation

$$\nabla^2 \psi - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0.$$

The electromagnetic wave speed is therefore

$$\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}.$$

In vacuum,

$$\mu = \mu_0, \quad \epsilon = \epsilon_0,$$

so

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 2.998 \times 10^8 \text{ m/s}.$$

This is one of the profound outcomes of Maxwell's theory: the speed predicted for electromagnetic disturbances is the measured speed of light. Light is therefore an electromagnetic wave.

What does $\nabla^2 \mathbf{E}$ mean?

The vector Laplacian acts on the spatial structure of the field. In Cartesian coordinates with constant unit vectors,

$$\nabla^2 \mathbf{E} = \hat{\mathbf{x}} \nabla^2 E_x + \hat{\mathbf{y}} \nabla^2 E_y + \hat{\mathbf{z}} \nabla^2 E_z.$$

The wave equation says, roughly, that **spatial curvature of the field is tied to temporal acceleration of the field**. That is the mathematical signature of propagation.

3. The magnetic-field wave equation and the symmetry of propagation

The magnetic-field equation follows by the same logic, beginning with the source-free Ampère-Maxwell law:

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Take the curl:

$$\nabla \times (\nabla \times \mathbf{H}) = \epsilon \frac{\partial}{\partial t} (\nabla \times \mathbf{E}).$$

Use Faraday's law,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$

to obtain

$$\nabla \times (\nabla \times \mathbf{H}) = -\mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

In a region with no magnetic monopoles,

$$\nabla \cdot \mathbf{B} = 0.$$

For uniform μ , this implies

$$\nabla \cdot \mathbf{H} = 0.$$

Applying the curl-of-curl identity gives

$$\boxed{\nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0}.$$

The electric and magnetic fields therefore propagate with the **same wave speed**. They are not two independent waves travelling together by coincidence. Maxwell's equations continuously lock their spatial and temporal behaviour together.

Transverse behaviour in a source-free plane wave

Consider a plane wave whose field varies only with z . If

$$\mathbf{E} = \hat{\mathbf{x}} E_x(z, t),$$

then the field is transverse to the propagation direction $\hat{\mathbf{z}}$. The associated magnetic field will be transverse to both:

$$\mathbf{H} \parallel \hat{\mathbf{y}}.$$

Thus, for a simple uniform plane wave,

$$\mathbf{E} \perp \mathbf{H}, \quad \mathbf{E} \perp \hat{\mathbf{k}}, \quad \mathbf{H} \perp \hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}$ indicates the direction of propagation.

The direction of power flow follows the right-handed triad

$$\mathbf{E} \times \mathbf{H} \parallel \hat{\mathbf{k}}.$$

We shall derive these relationships carefully in Document 10.

Why the divergence equations still matter

It is tempting to think that only the two curl equations are important for waves. That is incomplete. The divergence laws provide **constraints** on permissible fields. A numerical field that satisfies a curl equation but violates Gauss's laws can contain nonphysical components. Good EM formulations preserve or control these constraints.

Teaching point: the curl equations provide the dynamical coupling; the divergence equations restrict the field geometry and sources.

4. The one-dimensional travelling-wave solution

To see what the wave equation actually does, suppose one field component depends only on z and t :

$$E_x = E_x(z, t).$$

The vector wave equation reduces to

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2} = 0.$$

A general solution is the sum of a forward and backward travelling disturbance:

$$E_x(z, t) = F\left(t - \frac{z}{v}\right) + G\left(t + \frac{z}{v}\right).$$

The first term travels in the $+z$ direction. To see why, hold its argument constant:

$$t - \frac{z}{v} = \text{constant}.$$

As t increases, z must also increase at speed v . Similarly, $G(t + z/v)$ moves toward $-z$.

For a sinusoidal forward wave,

$$E_x(z, t) = E_0 \cos(\omega t - \beta z + \phi_0).$$

A point of constant phase satisfies

$$\omega t - \beta z + \phi_0 = \text{constant},$$

so

$$\frac{dz}{dt} = \frac{\omega}{\beta}.$$

The **phase velocity** is therefore

$$v_p = \frac{\omega}{\beta}.$$

One full spatial cycle corresponds to a phase change of 2π , hence

$$\beta = \frac{2\pi}{\lambda}.$$

Since $\omega = 2\pi f$,

$$v_p = f\lambda.$$

In a lossless homogeneous nondispersive medium,

$$v_p = \frac{1}{\sqrt{\mu\epsilon}},$$

so

$$\lambda = \text{rac} v_p f.$$

Worked example: wavelength in a dielectric

For a nonmagnetic material with relative permittivity $\epsilon_r = 4$,

$$\mu_r \approx 1,$$

and therefore

$$v \approx \frac{c}{\sqrt{\epsilon_r}} = \frac{c}{2}.$$

At 1 GHz,

$$\lambda_0 \approx 0.300 \text{ m},$$

while inside the material

$$\lambda \approx 0.150 \text{ m}.$$

The **frequency is unchanged** at a material boundary in steady state; the speed and wavelength change together.

Core mental model: ω describes how rapidly phase changes with time; β describes how rapidly it changes with distance.

5. Phasors turn the wave equation into the Helmholtz equation

For sinusoidal steady state using the course convention $e^{j\omega t}$, write

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\tilde{\mathbf{E}}(\mathbf{r})e^{j\omega t}\}.$$

Two time differentiations correspond to multiplication by

$$(j\omega)^2 = -\omega^2.$$

Therefore the time-domain wave equation

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

becomes

$$\boxed{\nabla^2 \tilde{\mathbf{E}} + k^2 \tilde{\mathbf{E}} = 0}$$

with

$$k = \omega\sqrt{\mu\epsilon}.$$

This is the **vector Helmholtz equation**. The same equation holds for $\tilde{\mathbf{H}}$ in the same homogeneous source-free medium.

The quantity k is the **wavenumber**, measured in radians per metre:

$$k = \frac{2\pi}{\lambda}.$$

For a uniform plane wave travelling in $+z$,

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}E_0 e^{-jkz}.$$

Restoring time dependence gives

$$\mathbf{E}(z, t) = \hat{\mathbf{x}}E_0 \cos(\omega t - kz),$$

which indeed travels in the $+z$ direction for the $e^{j\omega t}$ convention.

A backward wave uses

$$e^{+jkz}.$$

k and β

In an unbounded, homogeneous, lossless medium we often write

$$k = \beta.$$

In guided structures, however, it is useful to distinguish the material wavenumber k from the propagation constant along the guide, β . For a rectangular waveguide,

$$\beta = \sqrt{k^2 - k_c^2},$$

where k_c is set by the transverse boundary conditions. This distinction is why the guide wavelength differs from the free-space or material wavelength.

Helmholtz is everywhere in frequency-domain EM

Once sinusoidal steady state is assumed, many EM problems reduce to some form of the Helmholtz equation with suitable sources and boundary conditions. Analytical waveguide theory, scattering theory, Green's functions, FEM frequency-domain formulations and MoM formulations all contain this structure.

Solver connection: the Helmholtz operator is one of the main mathematical objects that computational EM discretises.

6. Lossy media: attenuation and the complex propagation constant

Real materials may conduct. In a homogeneous conducting medium,

$$\mathbf{J} = \sigma \mathbf{E},$$

so the phasor Ampère-Maxwell law becomes

$$\nabla \times \tilde{\mathbf{H}} = (\sigma + j\omega\epsilon)\tilde{\mathbf{E}}.$$

Repeating the wave-equation derivation gives

$$\nabla^2 \tilde{\mathbf{E}} + (\omega^2\mu\epsilon - j\omega\mu\sigma)\tilde{\mathbf{E}} = 0.$$

A wave travelling in +z is conveniently written

$$\tilde{\mathbf{E}}(z) = \mathbf{E}_0 e^{-\gamma z},$$

where

$$\boxed{\gamma = \alpha + j\beta}$$

is the **complex propagation constant**. Here

- α is the attenuation constant in Np/m;
- β is the phase constant in rad/m.

Substitution gives

$$\boxed{\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)}.$$

The spatial factor separates naturally:

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}.$$

The first factor reduces amplitude with distance; the second advances phase.

Another useful representation introduces **complex permittivity**:

$$\tilde{\epsilon} = \epsilon - j \frac{\sigma}{\omega}.$$

Then the lossy-medium Helmholtz equation has the same algebraic form as before, provided ϵ is replaced by $\tilde{\epsilon}$.

Loss tangent

The relative importance of conduction and displacement current is measured by

$$\tan \delta = \frac{\sigma}{\omega\epsilon}$$

for this simple conduction-loss model. If $\sigma \ll \omega\epsilon$, the medium behaves approximately as a low-loss dielectric. If $\sigma \gg \omega\epsilon$, it behaves as a good conductor.

Good conductor and skin depth

For a good conductor,

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}.$$

The **skin depth** is

$$\boxed{\delta_s = \frac{1}{\alpha} \approx \sqrt{\frac{2}{\omega\mu\sigma}}}.$$

After one skin depth, the field amplitude has fallen to $e^{-1} \approx 36.8\%$ of its surface value. This is why RF current becomes concentrated near conductor surfaces as frequency rises.

7. Propagation constant, intrinsic impedance and the E-H relationship

The wave equation tells us how a field varies through space, but Maxwell's first-order curl equations tell us how $\mathbf{E}$ and $\mathbf{H}$ are related. Consider a lossless plane wave travelling in $+z$ with

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}E_0 e^{-j\beta z}.$$

Faraday's law is

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}.$$

For this field, evaluating the curl gives a magnetic field in the $+y$ direction:

$$\tilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{E_0}{\eta} e^{-j\beta z},$$

where

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

is the **intrinsic impedance** of a lossless medium.

Equivalently,

$$\tilde{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{k}} \times \tilde{\mathbf{E}}.$$

In free space,

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.73 \, \Omega.$$

The units are ohms because E/H has units

$$\frac{\text{V/m}}{\text{A/m}} = \Omega.$$

This is **not** a lumped resistor sitting in space. It is a field ratio imposed by Maxwell's equations for a travelling wave.

For a lossy medium the intrinsic impedance becomes complex:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}.$$

Then $\mathbf{E}$ and $\mathbf{H}$ are generally not exactly in phase.

Four quantities that should not be confused

For an unbounded medium:

$$k = \omega\sqrt{\mu\epsilon}$$

describes spatial phase rate in a lossless material;

$$\gamma = \alpha + j\beta$$

describes attenuation and phase in a lossy material;

$$\eta$$

relates electric and magnetic field amplitudes;

and

$$v_p = \frac{\omega}{\beta}$$

describes phase velocity.

Keeping these roles separate makes later waveguide and transmission-line theory much easier.

8. Superposition, reflections and standing-wave structure

The wave equation is linear in a linear medium, so solutions can be added. In one dimension the phasor field can therefore contain both forward and backward waves:

$$\tilde{E}(z) = E_f e^{-j\beta z} + E_b e^{+j\beta z}.$$

The first travels toward $+z$ and the second toward $-z$ for our $e^{j\omega t}$ convention.

If the two have equal amplitudes,

$$E_f = E_b = E_0,$$

then

$$\tilde{E}(z) = E_0(e^{-j\beta z} + e^{j\beta z}) = 2E_0 \cos \beta z.$$

The spatial envelope now contains fixed nodes and antinodes. Restoring time dependence,

$$E(z, t) = 2E_0 \cos \beta z \cos \omega t.$$

This is a **standing wave**. It is not a third kind of electromagnetic propagation. It is the interference pattern formed by two counter-propagating travelling waves.

Boundary conditions select the allowed solution

The wave equation alone admits many possible fields. A physical problem becomes definite only when we also state:

- the geometry;
- material parameters;
- sources or excitation;
- boundary conditions;
- radiation conditions where the domain is open.

For example, a perfect electric conductor requires the tangential electric field at its surface to vanish. In a rectangular waveguide, applying this condition at all four conducting walls permits only particular transverse field patterns. Those permitted patterns are the TE and TM modes.

Thus the sequence is

Maxwell $\rightarrow$ wave/Helmholtz equation $\rightarrow$ boundary conditions $\rightarrow$ allowed fields

Why resonators have discrete frequencies

If waves are confined between boundaries, a field can reproduce itself only at certain frequencies. The allowed field patterns are eigenfunctions and the associated frequencies are eigenvalues. This is the origin of resonant modes in cavities and cutoff modes in waveguides.

Why open-space scattering is different

For an antenna or scattering problem, the domain is not closed. Instead of conducting walls surrounding everything, we impose an **outgoing-wave radiation condition**: energy must leave the source region rather than arrive from infinity unless an incident wave is explicitly specified.

Core mental model: the differential equation describes what fields may do locally; the boundary and radiation conditions decide which global solution is physically relevant.

9. Dispersion, phase velocity and group velocity

So far we have mostly considered a nondispersive homogeneous medium, where

$$\beta = \omega \sqrt{\mu\epsilon}.$$

If μ and ϵ are effectively constant with frequency, then

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}.$$

A modulated signal, however, contains a range of frequencies. The velocity at which the envelope of a narrowband group of frequencies moves is the **group velocity**:

$$v_g = \frac{d\omega}{d\beta}.$$

In a nondispersive unbounded medium,

$$v_g = v_p.$$

In a dispersive system, β is not proportional to ω , so phase and group velocities differ. Different spectral components accumulate different phase shifts, which can distort pulses or modulation envelopes.

Material dispersion

Real permittivity and permeability can depend on frequency:

$$\epsilon = \epsilon(\omega), \quad \mu = \mu(\omega).$$

Even an unbounded material can therefore be dispersive.

Waveguide dispersion

A hollow metallic waveguide is dispersive even when filled with an ideal nondispersive dielectric. For a mode with cutoff wavenumber k_c ,

$$\beta = \sqrt{k^2 - k_c^2}.$$

For an air-filled guide, this leads to

$$v_p = \frac{c}{\sqrt{1 - (f_c/f)^2}}$$

and

$$v_g = c\sqrt{1 - (f_c/f)^2}.$$

Hence

$$v_p v_g = c^2$$

for the ideal air-filled guide.

Above cutoff, v_p can exceed c without violating relativity because phase velocity is not the speed at which information or energy is transported. The group velocity is below c in this ideal case.

Connection to your waveguide calculations

The familiar guide wavelength

$$\lambda_g = \frac{2\pi}{\beta}$$

becomes very large as f approaches f_c from above because $\beta \rightarrow 0$. The field pattern then changes phase only slowly along the guide even though its time oscillation remains at frequency f .

This is a good example of why k , β , λ and λ_g must be kept conceptually separate.

10. Synthesis, solver connection and degree-level checkpoint

The chain developed in this document is one of the most important in electromagnetics:

$$\boxed{\text{Maxwell's equations} \Rightarrow \text{wave equation} \Rightarrow \text{Helmholtz equation} \Rightarrow \text{propagating field solutions}}$$

For a simple source-free, homogeneous, lossless medium,

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

and in sinusoidal steady state,

$$\nabla^2 \tilde{\mathbf{E}} + k^2 \tilde{\mathbf{E}} = 0, \quad k = \omega \sqrt{\mu\epsilon}.$$

The corresponding wave speed and wavelength are

$$v = \frac{1}{\sqrt{\mu\epsilon}}, \quad \lambda = \frac{2\pi}{k} = \frac{v}{f}.$$

In a lossy medium,

$$\gamma = \alpha + j\beta,$$

and the travelling-wave factor becomes

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z}.$$

How an EM solver uses this mathematics

A frequency-domain solver does not usually “simulate a wave” by drawing moving sinusoids. It solves a spatial field equation derived from Maxwell, typically a curl-curl or Helmholtz-type equation. Boundary conditions, material properties and sources define the problem. Discretisation then converts the continuous operator equation into a finite matrix system.

For example, an FEM formulation may lead schematically to

$$\mathbf{A}\mathbf{e} = \mathbf{b},$$

where the vector $\mathbf{e}$ contains unknown field coefficients. An eigenmode problem may instead become

$$\mathbf{K}\mathbf{e} = \omega^2 \mathbf{M}\mathbf{e}.$$

MoM starts from an equivalent integral equation, but the underlying Green’s function is itself a solution of a Helmholtz-type differential equation. The same physics keeps reappearing in different mathematical forms.

Degree-level checkpoint

You should now be able to answer or derive the following without relying on memorised slogans:

1. Starting from Maxwell’s equations, derive the electric-field wave equation in a source-free homogeneous lossless medium.
2. Explain exactly where the assumption $\nabla \cdot \mathbf{E} = 0$ enters the derivation.
3. Derive the corresponding wave equation for $\mathbf{H}$.
4. Show that $F(t - z/v)$ satisfies the one-dimensional wave equation.
5. Explain physically what ω , k , β , λ , α and γ each describe.
6. Convert the time-domain wave equation into the Helmholtz equation using the $e^{j\omega t}$ convention.
7. Explain why $e^{-j\beta z}$ represents propagation in $+z$ with that convention.
8. Explain how conductivity changes a travelling wave and define skin depth.
9. Distinguish phase velocity from group velocity and explain why $v_p > c$ in a waveguide is not superluminal information transfer.
10. Explain why boundary conditions are required in addition to the wave equation.

Teach-back test: if you can derive the wave equation on a board, state every assumption as you use it, and explain the physical meaning of each term, you have moved from remembering EM formulas to understanding the structure of electromagnetic wave theory.