

Gradient, Divergence and Curl

Electromagnetic Theory Course — Maxwell and Wave Theory — Document 6 of 23

1. Why electromagnetics needs differential operators

Electromagnetic fields vary from point to point. Knowing the value of a field at one location is therefore only part of the story; we also need to know **how that field changes locally**. Three operations do most of this work:

$$\nabla f, \quad \nabla \cdot \mathbf{A}, \quad \nabla \times \mathbf{A}.$$

They are called the **gradient**, **divergence** and **curl**. They are built from the differential operator

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z},$$

pronounced *del* or *nabla*. In Cartesian coordinates it looks superficially like a vector, but it is better thought of as an **instruction to differentiate what follows it**.

The three operators answer three different physical questions.

Operator	Input	Output	Physical question
∇f	scalar field	vector field	In which direction does f increase fastest?
$\nabla \cdot \mathbf{A}$	vector field	scalar field	Is there a net local outflow or inflow?
$\nabla \times \mathbf{A}$	vector field	vector field	Is there local circulation, and about which axis?

These are not merely mathematical decorations. Maxwell's equations are built from divergence and curl:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_f, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \end{aligned}$$

The first pair describe **sources and flux**. The second pair describe **circulation and induction**.

Local versus global descriptions

A differential equation describes what is happening in an infinitesimal neighbourhood around each point. An integral equation describes the accumulated behaviour over a finite line, surface or volume. The divergence theorem and Stokes' theorem connect these two views.

By the end of this document you should be able to:

- calculate grad, div and curl in Cartesian coordinates;
- explain each operator physically, not just symbolically;
- relate divergence to flux and curl to circulation;
- use Gauss's and Stokes' theorems to move between local and integral descriptions;
- recognise the operators in Maxwell, waveguide and numerical-solver equations.

Core mental model: gradient concerns the *slope of a scalar field*; divergence concerns the *net expansion of a vector field*; curl concerns the *local rotation of a vector field*.

2. Gradient: the direction of steepest increase

Let $f(x, y, z)$ be a scalar field. Its gradient is

$$\nabla f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}.$$

The result is a **vector**. It points in the direction in which f increases most rapidly, and its magnitude gives the maximum rate of increase per unit distance.

Suppose

$$f(x, y) = 3x^2 + 2y^2.$$

Then

$$\nabla f = 6x\hat{x} + 4y\hat{y}.$$

At $(x, y) = (1, 2)$,

$$\nabla f = 6\hat{x} + 8\hat{y}.$$

Its magnitude is

$$|\nabla f| = \sqrt{6^2 + 8^2} = 10.$$

So, at that point, the direction of fastest increase is along the unit vector

$$\hat{u} = \frac{\nabla f}{|\nabla f|} = 0.6\hat{x} + 0.8\hat{y}.$$

Directional derivative

If we move in some unit direction $\hat{u}$ rather than the steepest direction, the rate of change is

$$\frac{df}{ds} = \nabla f \cdot \hat{u}.$$

The dot product selects the component of the gradient along the chosen direction.

Electric potential gives the electric field

Electrostatics provides the most important EM example:

$$\mathbf{E} = -\nabla V.$$

If

$$V(x, y, z) = V_0 - ax,$$

then

$$\nabla V = -a\hat{x},$$

and therefore

$$\mathbf{E} = a\hat{x}.$$

The minus sign tells us that the electric field points from higher potential toward lower potential.

Equipotential surfaces

A surface on which V is constant is an **equipotential surface**. Moving tangentially along it produces no change in V , so

$$\nabla V \cdot d\mathbf{l} = 0$$

for any infinitesimal displacement tangent to that surface. Hence ∇V is normal to an equipotential surface, and therefore $\mathbf{E} = -\nabla V$ is normal to it as well.

This is why electrostatic field lines cross equipotential surfaces at right angles.

Teaching test: if you can explain why $\mathbf{E}$ must be perpendicular to a conducting surface in electrostatic equilibrium using $\mathbf{E} = -\nabla V$, you understand gradient physically rather than just algebraically.

3. Divergence: local sources, sinks and flux

For a vector field

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}},$$

the divergence in Cartesian coordinates is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

The result is a **scalar**. It measures the net outward flux from an infinitesimally small volume around a point, per unit volume.

Imagine a tiny closed box placed in a flowing vector field. If more field leaves the box than enters it, divergence is positive. If more enters than leaves, divergence is negative. If inflow and outflow balance, divergence is zero.

Worked field

Consider

$$\mathbf{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

Then

$$\nabla \cdot \mathbf{A} = 1 + 1 + 1 = 3.$$

The field points radially outward and becomes stronger with distance. Every small volume behaves like a local source, so positive divergence is exactly what we expect.

Now consider

$$\mathbf{A} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}.$$

Then

$$\nabla \cdot \mathbf{A} = 0.$$

This field circulates around the z axis but has no net source or sink. Circulation and divergence are different properties.

Divergence in electrostatics

Gauss's law in differential form is

$$\nabla \cdot \mathbf{D} = \rho_f.$$

This says that **free charge density is the local source of electric flux density**. Where $\rho_f > 0$, net $\mathbf{D}$ flux emerges; where $\rho_f < 0$, net flux converges.

In a charge-free region,

$$\rho_f = 0 \Rightarrow \nabla \cdot \mathbf{D} = 0.$$

This does **not** mean $\mathbf{D} = 0$. It means that the local net flux leaving a tiny volume is zero.

Magnetic field

Maxwell's magnetic-flux law is

$$\nabla \cdot \mathbf{B} = 0.$$

Magnetic flux has no observed isolated source or sink analogous to electric charge. Magnetic field lines therefore form closed loops or extend through the boundary of the region without beginning or ending there.

Common mistake: zero divergence does not mean zero field. A uniform field has zero divergence, and a circulating field may also have zero divergence.

4. Curl: local circulation and rotation

Curl measures the local circulation of a vector field. For

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}},$$

the Cartesian curl is found component by component:

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z},$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

Therefore,

$$\nabla \times \mathbf{A} = (\nabla \times \mathbf{A})_x \hat{\mathbf{x}} + (\nabla \times \mathbf{A})_y \hat{\mathbf{y}} + (\nabla \times \mathbf{A})_z \hat{\mathbf{z}}.$$

The direction of $\nabla \times \mathbf{A}$ gives the axis of local circulation according to the right-hand rule.

A field that rotates

Take

$$\mathbf{A} = -y \hat{\mathbf{x}} + x \hat{\mathbf{y}}.$$

Its curl is

$$\nabla \times \mathbf{A} = 2 \hat{\mathbf{z}}.$$

The vectors circulate anticlockwise when viewed from +z, so a curl in the +z direction is physically sensible.

Curl in Maxwell's equations

Faraday's law says

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

A time-changing magnetic flux density therefore creates a **circulating electric field**.

Ampère-Maxwell says

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Conduction current and changing electric flux density create a circulating magnetic field.

This is the mathematical engine behind electromagnetic-wave propagation: a changing magnetic field is associated with curl of $\mathbf{E}$, while a changing electric field is associated with curl of $\mathbf{H}$.

Curl is not the same as curved field lines

A vector field can have curved-looking field lines yet have zero curl, and a field can have nonzero curl without looking like rigid rotation over a large region. Curl is a **local differential measure**, not simply a visual judgement about whether a plotted line bends.

Physical image: imagine placing a tiny freely rotating paddle wheel into the field. Curl describes the infinitesimal tendency for that wheel to rotate, including the axis and sense of rotation.

5. Divergence theorem and Stokes' theorem: local becomes global

Two major theorems connect differential operators to measurable integrals over finite regions.

Divergence theorem

For a volume V bounded by a closed surface S ,

$$\iiint_V (\nabla \cdot \mathbf{A}) \, dV = \oint_S \mathbf{A} \cdot d\mathbf{S}.$$

The left side adds the local divergence throughout the volume. The right side measures the total outward flux through the boundary. Thus:

total source strength inside = net flux through the outside surface.

Applying this to Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_f,$$

gives

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho_f \, dV = Q_{f,\text{enc}}$$

The familiar integral form and the differential form are therefore two descriptions of the same physical law.

Stokes' theorem

For an open surface S bounded by a closed contour C ,

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}.$$

The left side accumulates local curl across the surface. The right side measures circulation around its boundary.

Apply this to Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Integrating over S and using Stokes gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

This is the transformer-emf form of Faraday's law.

Orientation matters

The surface normal $d\mathbf{S}$ and the positive direction around C are connected by the **right-hand rule**. Reverse the surface normal and the positive contour direction also reverses.

Why these theorems matter in EM

They let us move between two useful viewpoints:

- **integral form** — excellent for symmetry, physical interpretation and measurements;
- **differential form** — excellent for local field equations, PDEs and numerical solvers.

Teaching insight: Gauss's and Stokes' theorems do not add new physics to Maxwell's equations. They provide the mathematical bridge that shows why the integral and differential versions encode the same physics under appropriate smoothness conditions.

6. Gradient, divergence and curl in useful coordinate systems

Cartesian formulas are simplest, but EM geometry often suggests cylindrical or spherical coordinates. The physics does not change; the coordinate scale factors do.

Cylindrical coordinates (ρ, ϕ, z)

For a scalar f ,

$$\nabla f = \hat{\rho} \frac{\partial f}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \hat{z} \frac{\partial f}{\partial z}.$$

For

$$\begin{aligned} \mathbf{A} &= A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}, \\ \nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \end{aligned}$$

and its curl components are

$$\begin{aligned} (\nabla \times \mathbf{A})_\rho &= \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}, \\ (\nabla \times \mathbf{A})_\phi &= \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}, \\ (\nabla \times \mathbf{A})_z &= \frac{1}{\rho} \left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right]. \end{aligned}$$

Cylindrical coordinates naturally describe coaxial lines, wires, circular waveguides and many antenna structures.

Spherical coordinates (r, θ, ϕ)

The gradient is

$$\nabla f = \hat{r} \frac{\partial f}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}.$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}.$$

The extra r , ρ and $\sin \theta$ factors arise because the coordinate basis and physical cell sizes vary with position.

Practical advice: understand where the scale factors come from, but do not waste effort memorising every curvilinear curl component. In professional work they are normally checked from a trusted reference. What you must understand is which operator the physics requires and how symmetry removes terms.

7. Reading Maxwell's equations through grad, div and curl

Once the operators have physical meaning, Maxwell's differential equations become much less mysterious.

Gauss's law for electricity

$$\nabla \cdot \mathbf{D} = \rho_f.$$

Read it as:

free charge is the local source or sink of electric flux density.

If a small volume contains positive free charge, more $\mathbf{D}$ flux leaves than enters.

Gauss's law for magnetism

$$\nabla \cdot \mathbf{B} = 0.$$

Read it as:

magnetic flux has no local net source or sink.

This is why a correct numerical magnetic-field solution should conserve magnetic flux locally, apart from discretisation error.

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Read it as:

a changing magnetic flux density is associated with circulating electric field.

In phasor form with the $e^{j\omega t}$ convention,

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}.$$

Ampère-Maxwell law

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Read it as:

free current and changing electric flux density are associated with circulating magnetic field.

In a source-free, homogeneous region,

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}}.$$

Together these two curl equations allow one time-varying field to sustain the other. Taking another curl is the route to the electromagnetic wave equation.

Where gradient enters

Gradient appears explicitly when a field is represented by a potential. In electrostatics,

$$\mathbf{E} = -\nabla V.$$

In full electromagnetic theory, potentials are related by

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.\end{aligned}$$

So the three operators are not separate tricks: they form a connected language for sources, potentials, circulation and waves.

A useful habit: when you see $\nabla \cdot$, say “net local flux”; when you see $\nabla \times$, say “local circulation”; when you see ∇ acting on a scalar, say “direction and rate of steepest increase.”

8. Vector identities, the Laplacian and why they matter

A few vector-calculus identities repeatedly simplify electromagnetic derivations.

Curl of a gradient

For a sufficiently smooth scalar field f ,

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Therefore, if

$$\mathbf{E} = -\nabla V,$$

then automatically

$$\nabla \times \mathbf{E} = \mathbf{0}$$

in electrostatics. This is the mathematical statement that an electrostatic field is conservative.

Divergence of a curl

For a sufficiently smooth vector field $\mathbf{A}$,

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

Thus defining

$$\mathbf{B} = \nabla \times \mathbf{A}$$

automatically satisfies

$$\nabla \cdot \mathbf{B} = 0.$$

That is one reason the magnetic vector potential is so useful.

Scalar Laplacian

The Laplacian of a scalar field is

$$\nabla^2 f = \nabla \cdot (\nabla f).$$

In Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Electrostatic potential in a charge-free homogeneous region satisfies Laplace's equation,

$$\nabla^2 V = 0,$$

while in a region containing charge,

$$\nabla^2 V = -\frac{\rho}{\epsilon},$$

which is Poisson's equation.

Curl-curl identity

A central identity is

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

In a homogeneous source-free region where $\nabla \cdot \mathbf{E} = 0$, this reduces the double curl of $\mathbf{E}$ to $-\nabla^2 \mathbf{E}$. Applying curl to Maxwell's equations then produces the wave equation.

These identities are therefore not abstract exercises. They explain why potentials can enforce Maxwell constraints and why wave equations emerge from the two curl laws.

Common trap: $\nabla^2 \mathbf{A}$ is a vector Laplacian, whereas $\nabla^2 f$ is scalar. In Cartesian coordinates the vector Laplacian can be applied component by component; in curvilinear coordinates extra terms appear.

9. What grad, div and curl become inside EM solvers

A numerical EM solver replaces continuous differential operations by algebraic operations on a finite set of unknowns. The underlying physics, however, is still grad, div and curl.

Finite differences

A derivative such as

$$\frac{\partial E_y}{\partial x}$$

may be approximated on a grid by

$$\left. \frac{\partial E_y}{\partial x} \right|_i \approx \frac{E_{y,i+1} - E_{y,i}}{\Delta x}.$$

Curl is then built from combinations of such differences. FDTD does this directly when it advances Maxwell's curl equations in time. On the Yee grid, electric- and magnetic-field components are deliberately **staggered in space**. This geometry makes the discrete curl operation natural and helps preserve important conservation properties.

Finite elements

FEM usually starts from a differential equation containing operators such as

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \epsilon \mathbf{E} = -j\omega \mathbf{J}.$$

After choosing basis functions and applying a weak formulation, the continuous operator equation becomes a sparse matrix system such as

$$\mathbf{K}\mathbf{e} - \omega^2 \mathbf{M}\mathbf{e} = \mathbf{f}.$$

The matrices encode integrals involving the curls, gradients and material properties of the basis functions.

Method of Moments

A surface-integral MoM formulation may hide differential operators inside a Green's function or integral kernel. Even then the Green's function itself came from a differential equation such as

$$(\nabla^2 + k^2)G = -\delta.$$

So MoM does not escape vector calculus; it rearranges the field problem into an integral equation before discretisation.

Numerical consistency matters

If a numerical method badly violates identities such as

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0,$$

it can generate nonphysical solutions or spurious modes. Modern computational EM methods therefore pay close attention to the discrete equivalents of grad, div and curl.

Solver mental model: the differential operators describe local physics continuously; discretisation turns those operators into matrices or finite-difference stencils that a computer can evaluate.

10. Synthesis, common mistakes and degree-level checkpoint

The three operators can now be summarised compactly:

Operation	Mapping	Meaning
∇f	scalar $\rightarrow$ vector	direction and rate of steepest increase
$\nabla \cdot \mathbf{A}$	vector $\rightarrow$ scalar	local net outflow or inflow
$\nabla \times \mathbf{A}$	vector $\rightarrow$ vector	local circulation and its axis

A synthesis example

Consider

$$V(x, y, z) = V_0 - ax,$$

so

$$\mathbf{E} = -\nabla V = a\hat{\mathbf{x}}.$$

Because the field is uniform,

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = \mathbf{0}.$$

This is entirely consistent: there is a nonzero electric field, but there is no local charge within the region and no electro static circulation. A field can therefore be nonzero while both its divergence and curl are zero.

Common mistakes to eliminate

1. Treating ∇ as an ordinary vector rather than a differential operator.
2. Forgetting that gradient acts naturally on a scalar and returns a vector.
3. Assuming zero divergence means zero field.
4. Assuming curved field lines automatically mean nonzero curl.
5. Ignoring surface orientation or contour direction in Stokes' theorem.
6. Using Cartesian formulas unchanged in cylindrical or spherical coordinates.
7. Confusing an integral statement over a finite region with a differential statement at one point.
8. Forgetting that partial derivatives hold the other independent variables fixed.

Degree-level checkpoint

1. For $f = x^2y + 3z$, calculate ∇f .
2. For $\mathbf{A} = x^2\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} - z\hat{\mathbf{z}}$, calculate $\nabla \cdot \mathbf{A}$.
3. For $\mathbf{A} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, calculate $\nabla \times \mathbf{A}$ and explain its direction physically.
4. Explain in words why $\nabla \cdot \mathbf{B} = 0$ does not imply $\mathbf{B} = \mathbf{0}$.
5. Starting from $\mathbf{E} = -\nabla V$, show that an electrostatic field has zero curl.
6. State the divergence theorem and explain how it converts differential Gauss law into integral Gauss law.
7. State Stokes' theorem and explain its role in converting Faraday's differential law into its contour-integral form.
8. Explain why an FDTD solver needs a discrete approximation to curl but a conventional driven MoM solver may hide curl inside its integral formulation.

Short answers

For Question 1,

$$\nabla f = 2xy\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}.$$

For Question 2,

$$\nabla \cdot \mathbf{A} = 2x + 2 - 1 = 2x + 1.$$

For Question 3,

$$\nabla \times \mathbf{A} = 2\hat{\mathbf{z}}.$$

The remaining questions are deliberately explanatory. If you can answer them aloud without relying on memorised wording, you are ready to use these operators confidently in Maxwell's equations.

Teach-back target: explain grad, div and curl to a first-year engineering student using one physical picture, one equation and one EM example for each. If the student can predict the sign or direction before doing the algebra, your explanation has succeeded.