

Multivariable Calculus and Integration for Electromagnetics

Electromagnetic Theory Course — Mathematical Foundations — Document 4 of 23

1. Functions of several variables and partial differentiation

Electromagnetic fields normally depend on more than one independent variable. A temperature along a wire might be written as $T(x)$, but an electric field in space may depend on three spatial coordinates and on time:

$$\mathbf{E} = \mathbf{E}(x, y, z, t).$$

Each component is itself a function. For example,

$$\mathbf{E} = E_x(x, y, z, t)\hat{\mathbf{x}} + E_y(x, y, z, t)\hat{\mathbf{y}} + E_z(x, y, z, t)\hat{\mathbf{z}}.$$

A **partial derivative** asks how a multivariable function changes when one variable changes while the others are temporarily held fixed. If

$$f(x, y) = x^2y + 3y^2,$$

then differentiation with respect to x treats y as a constant:

$$\frac{\partial f}{\partial x} = 2xy.$$

Differentiation with respect to y instead gives

$$\frac{\partial f}{\partial y} = x^2 + 6y.$$

The symbol ∂ rather than d warns us that the function has more than one independent variable.

Why this matters in EM

Suppose

$$E_x(x, z) = E_0 \sin\left(\frac{\pi x}{a}\right) \cos(\beta z).$$

Then

$$\frac{\partial E_x}{\partial x} = E_0 \frac{\pi}{a} \cos\left(\frac{\pi x}{a}\right) \cos(\beta z),$$

whereas

$$\frac{\partial E_x}{\partial z} = -E_0 \beta \sin\left(\frac{\pi x}{a}\right) \sin(\beta z).$$

The first describes how the field varies **across** the guide; the second describes how it varies **along** the guide. The same field can vary differently in different directions.

Second partial derivatives are written, for example,

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}.$$

For sufficiently smooth functions, the mixed derivatives can usually be interchanged:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

This smoothness assumption is common in ordinary EM regions, though singular sources and material discontinuities require greater care.

Teaching idea: a partial derivative is a local slope measured in one chosen direction while all other coordinates are frozen. Maxwell's differential equations are built from such local directional changes.

2. Total change, differentials and the chain rule

A partial derivative isolates one variable, but a real path through space can change several coordinates at once. For a scalar function

$$f = f(x, y, z),$$

a small total change is approximated by the **total differential**

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Each term says: local slope in one coordinate direction multiplied by the small displacement in that direction.

Suppose

$$V(x, y) = 3x^2 + 2xy.$$

Then

$$\frac{\partial V}{\partial x} = 6x + 2y, \quad \frac{\partial V}{\partial y} = 2x,$$

so

$$dV = (6x + 2y)dx + 2x dy.$$

This is useful in electrostatics because voltage is a scalar field and electric field is related to how that voltage changes in space.

The chain rule

If position itself depends on another variable, such as time, then a spatial field becomes a function of time **through the motion of the observation point**. If

$$f = f(x(t), y(t), z(t)),$$

then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$

If the field also has explicit time dependence, $f = f(x, y, z, t)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$

This distinction between **partial** and **total** time derivatives appears throughout field theory, moving media and continuum mechanics.

Product and quotient rules still apply

For functions of several variables, ordinary differentiation rules remain valid while differentiating with respect to the chosen variable. For example,

$$\frac{\partial}{\partial x}(fg) = f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}.$$

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2}.$$

Dimensions provide a useful check

If V is measured in volts and x in metres, then

$$\frac{\partial V}{\partial x}$$

has units V/m. That is exactly the unit of electric field. Dimensional checks are an excellent way to detect a missing length factor or an incorrect derivative.

Common mistake: $\partial f / \partial x$ does not mean that only x exists. It means that x is the variable being changed for this particular derivative; the other independent variables are being held fixed during the operation.

3. Indefinite and definite integration

Differentiation asks for a local rate of change. Integration performs the complementary operation: it **accumulates** contributions.

An **indefinite integral** is a family of antiderivatives:

$$\int f(x) dx = F(x) + C,$$

where

$$\frac{dF}{dx} = f(x).$$

The constant C is necessary because differentiation removes constants. For example,

$$\int 2x dx = x^2 + C.$$

An **indefinite** integral therefore does not produce one numerical answer; it produces a general function.

A **definite integral** accumulates between specified limits:

$$\int_a^b f(x) dx.$$

The Fundamental Theorem of Calculus gives

$$\int_a^b f(x) dx = F(b) - F(a).$$

For example,

$$\int_0^2 2x dx = [x^2]_0^2 = 4.$$

Physical interpretation

Think of integration as “sum many infinitesimal contributions.” If a line charge density $\lambda(x)$ has units C/m, then a small piece dx contains charge

$$dQ = \lambda(x) dx.$$

The charge between $x = a$ and $x = b$ is

$$Q = \int_a^b \lambda(x) dx.$$

Units make the accumulation clear:

$$\frac{\text{C}}{\text{m}} \times \text{m} = \text{C}.$$

If the density is uniform, $\lambda = \lambda_0$, then

$$Q = \lambda_0(b - a).$$

Average value

The average of a function over an interval of length $b - a$ is

$$\langle f \rangle = \frac{1}{b - a} \int_a^b f(x) dx.$$

This idea reappears in RF when calculating average power over a period or average field quantities over a region.

Integration is signed

A definite integral is not simply “area.” Contributions below the axis are negative because the integral keeps track of sign. Likewise, electromagnetic flux integrals keep track of **orientation**, not merely magnitude.

Teaching idea: differentiation is local; integration is cumulative. Maxwell’s differential equations tell you what happens at a point, whereas Maxwell’s integral equations accumulate field behaviour along paths or across surfaces.

4. Double integrals and integration over an area

When a quantity is distributed over a two-dimensional region, a single integral is not enough. A **double integral** sums contributions over area:

$$\iint_A f(x, y) dA.$$

In Cartesian coordinates,

$$dA = dx dy,$$

so over a rectangular region $a \leq x \leq b, c \leq y \leq d$,

$$\iint_A f dA = \int_c^d \int_a^b f(x, y) dx dy.$$

The inner integral is performed first. Its result is then integrated over the second variable.

Example: charge on a sheet

Let the surface charge density be

$$\rho_s(x, y) = \rho_0 \frac{x}{a}$$

over $0 \leq x \leq a, 0 \leq y \leq b$. The total charge is

$$Q = \iint_A \rho_s dA = \int_0^b \int_0^a \rho_0 \frac{x}{a} dx dy.$$

Evaluate the x integral:

$$\int_0^a \rho_0 \frac{x}{a} dx = \frac{\rho_0 a}{2}.$$

Then integrate over y :

$$Q = \frac{\rho_0 ab}{2}.$$

Since ρ_s has units of coulombs per square metre and dA has units of square metres, the answer has units C.

Why order may matter computationally

For well-behaved functions over ordinary regions, changing the order of integration gives the same result, but one order may be far easier. For irregular boundaries, the limits can depend on the outer variable:

$$\int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} f(x, y) dx dy.$$

This is not just mathematical bookkeeping. In EM, geometry determines limits: a rectangular aperture, circular conductor or triangular finite element all lead to different integration domains.

Area integrals versus surface flux integrals

A scalar area integral such as

$$\iint_A \rho_s dA$$

adds a scalar density. A flux integral such as

$$\iint_S \mathbf{D} \cdot d\mathbf{S}$$

also involves direction and orientation. We treat that distinction explicitly on page 7.

Common mistake: dA is a scalar area element. The vector surface element $d\mathbf{S} = \hat{\mathbf{n}} dS$ contains both area and orientation.

5. Line integrals: accumulating along a path

A **line integral** accumulates a quantity along a curve C . There are two important forms.

A scalar line integral is

$$\int_C f \, dl,$$

where dl is a small scalar length. It adds a scalar quantity distributed along a path.

A vector line integral is

$$\int_C \mathbf{A} \cdot d\mathbf{l}.$$

Here

$$d\mathbf{l} = \hat{\mathbf{l}} \, dl$$

is a directed infinitesimal displacement tangent to the path. The dot product extracts the component of $\mathbf{A}$ along the direction of travel.

Electric potential difference

Electrostatics provides a central example:

$$V(B) - V(A) = - \int_A^B \mathbf{E} \cdot d\mathbf{l}.$$

If

$$\mathbf{E} = E_0 \hat{\mathbf{x}}$$

and we move from $x = 0$ to $x = L$ along the x axis, then

$$d\mathbf{l} = \hat{\mathbf{x}} \, dx,$$

so

$$\mathbf{E} \cdot d\mathbf{l} = E_0 \, dx,$$

and

$$V(L) - V(0) = -E_0 L.$$

If instead the path moves entirely in the y direction while the field points in x , then

$$\mathbf{E} \cdot d\mathbf{l} = 0.$$

No potential change occurs because the displacement is perpendicular to the field.

Parameterising a curved path

A curve can be written as

$$\mathbf{r} = \mathbf{r}(u).$$

Then

$$d\mathbf{l} = \frac{d\mathbf{r}}{du} \, du,$$

and

$$\int_C \mathbf{A} \cdot d\mathbf{l} = \int_{u_1}^{u_2} \mathbf{A}(\mathbf{r}(u)) \cdot \frac{d\mathbf{r}}{du} \, du.$$

This converts a geometric path integral into an ordinary one-variable integral.

Orientation matters

Reversing a path reverses the sign:

$$\int_A^B \mathbf{A} \cdot d\mathbf{l} = - \int_B^A \mathbf{A} \cdot d\mathbf{l}.$$

That sign is physical. In circuit language, voltage rise versus voltage drop depends on direction; in field theory, circulation depends on the chosen contour orientation.

Physical picture: imagine walking along the path. At each point, ask “how much of this vector points in my direction of travel?” Add those contributions all the way along the curve.

6. Closed line integrals and circulation

A circle on the integral sign indicates a **closed path**:

$$\oint_C \mathbf{A} \cdot d\mathbf{l}.$$

The path starts and ends at the same point. Such an integral measures the accumulated tangential component of the field around a complete contour and is often called **circulation**.

Two of Maxwell's equations naturally use closed line integrals. Faraday's law is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S},$$

and the Ampere-Maxwell law is

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} + \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{S}.$$

At this stage, focus on the geometry rather than memorising the laws. The **left-hand side walks around a boundary**; the **right-hand side refers to what passes through the surface bounded by that contour**.

Simple circular example

Suppose a field around a circular path of radius a is everywhere tangent and has constant magnitude H_0 . Then

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_C H_0 dl.$$

The circumference is $2\pi a$, so

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = 2\pi a H_0.$$

If the field were normal to the path at every point instead, the dot product would vanish.

Choosing positive orientation

A closed contour must have an orientation. Once a surface normal is chosen, the positive circulation direction is related to it by the right-hand rule: curl the fingers of your right hand in the positive contour direction and your thumb points along the positive surface normal.

This pairing becomes essential in Stokes' theorem and in Faraday's law. Arbitrarily reversing either the contour or the normal without reversing the other creates a sign error.

Conservative fields

For an electrostatic field,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0.$$

That is why electrostatic potential is single-valued and path-independent. Time-varying magnetic flux changes the situation: Faraday's law tells us that a time-varying magnetic field can produce non-zero electric-field circulation.

Common mistake: "closed integral" is not a new integration technique by itself. The circle means the integration domain is closed: a loop for a line integral or a completely enclosing surface for a surface integral.

7. Surface integrals, vector area and electromagnetic flux

A **surface integral** accumulates over a two-dimensional surface embedded in three-dimensional space. For electromagnetic flux, the basic form is

$$\iint_S \mathbf{A} \cdot d\mathbf{S}.$$

The vector area element is

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the chosen unit normal to the surface. Therefore

$$\mathbf{A} \cdot d\mathbf{S} = A \cos\theta dS.$$

Only the component of $\mathbf{A}$ passing **through** the surface contributes.

Electric flux

Electric flux in terms of electric flux density is

$$\Psi_E = \iint_S \mathbf{D} \cdot d\mathbf{S}.$$

If $\mathbf{D}$ is uniform and perpendicular to a flat surface of area A , then

$$\Psi_E = DA.$$

If the surface is tilted by angle θ relative to the field direction,

$$\Psi_E = DA \cos\theta.$$

If the field lies entirely in the plane of the surface, $\theta = 90^\circ$ and the flux is zero.

Power through a surface

The same geometry appears in electromagnetic power. If $\mathbf{S}$ is the Poynting vector, then power through a surface is

$$P = \iint_A \mathbf{S} \cdot d\mathbf{S}.$$

The units confirm the interpretation:

$$\frac{\text{W}}{\text{m}^2} \times \text{m}^2 = \text{W}.$$

For waveguides, this integral is taken over the guide cross-section to obtain transported power.

Curved surfaces

For a curved surface, $\hat{\mathbf{n}}$ changes from point to point. The integral must therefore account for both the changing field and the changing local orientation. A sphere is a natural example: its outward normal is radial everywhere, even though the direction of that radial unit vector changes over the surface.

Scalar versus vector surface integrals

Do not confuse

$$\iint_S f dS$$

with

$$\iint_S \mathbf{A} \cdot d\mathbf{S}.$$

The first accumulates a scalar per unit area; the second calculates oriented flux of a vector field.

Physical picture: imagine the surface as a net. A flux integral asks how much of the vector field pierces the net, counting direction as positive or negative according to the chosen normal.

8. Closed surfaces and volume integrals

A **closed surface** completely encloses a volume. Its flux integral is often written

$$\oiint_S \mathbf{A} \cdot d\mathbf{S},$$

where the conventional positive normal points **outward** from the enclosed volume.

Gauss's law for the electric field uses exactly this construction:

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}.$$

The left-hand side asks for net outward electric flux through the enclosing surface. The right-hand side asks how much free charge is inside.

Magnetism has the corresponding relation

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0,$$

which expresses the absence of isolated magnetic monopoles in ordinary classical electromagnetism.

Volume integration

If a quantity is distributed throughout three-dimensional space, we use a triple or **volume integral**:

$$\iiint_V f \, dV.$$

For volume charge density ρ_v in coulombs per cubic metre,

$$Q = \iiint_V \rho_v \, dV.$$

For a uniform charge density in a rectangular box of dimensions a, b, c ,

$$Q = \int_0^c \int_0^b \int_0^a \rho_v \, dx \, dy \, dz = \rho_v abc.$$

Again the units tell the story:

$$\frac{\text{C}}{\text{m}^3} \times \text{m}^3 = \text{C}.$$

Energy stored in a field

Volume integration also appears in field energy. In a simple linear dielectric, electric energy density is

$$w_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \quad [\text{J}/\text{m}^3],$$

so the energy in a volume is

$$W_e = \iiint_V w_e \, dV.$$

This is an important conceptual shift: energy is not necessarily “located in a component.” Electromagnetic theory treats energy as distributed throughout the field in space.

Closed surface versus volume

A closed surface integral concerns what crosses the **boundary** of a volume. A volume integral concerns what exists **inside** it. The divergence theorem, introduced on page 10, is the mathematical bridge between these two descriptions.

Solver connection: finite-element and finite-difference methods repeatedly approximate area and volume integrals over small cells or elements. The continuous integral becomes a weighted sum over discrete pieces of geometry.

9. Integration in Cartesian, cylindrical and spherical coordinates

The meaning of an integral is coordinate-independent, but the differential length, area and volume elements depend on the coordinate system.

In Cartesian coordinates,

$$dV = dx dy dz.$$

In cylindrical coordinates (ρ, ϕ, z) ,

$$dV = \rho d\rho d\phi dz.$$

The factor ρ is essential. A small angular step $d\phi$ corresponds to an arc length $\rho d\phi$, which grows with radius.

In spherical coordinates (r, θ, ϕ) ,

$$dV = r^2 \sin\theta dr d\theta d\phi.$$

The factor $r^2 \sin\theta$ accounts for how coordinate spacing expands with radius and polar angle.

These scale factors are examples of a **Jacobian**: a factor that corrects the differential element when variables or coordinates are transformed.

Example: volume of a cylinder

For radius a and height L ,

$$V = \int_0^L \int_0^{2\pi} \int_0^a \rho d\rho d\phi dz.$$

Evaluate in sequence:

$$\int_0^a \rho d\rho = \frac{a^2}{2},$$

$$\int_0^{2\pi} d\phi = 2\pi,$$

so

$$V = \pi a^2 L.$$

Without the factor ρ , the dimensions and numerical result would both be wrong.

Example: surface area of a sphere

On a sphere of radius R , the radial surface element is

$$dS = R^2 \sin\theta d\theta d\phi.$$

Therefore

$$A = \int_0^{2\pi} \int_0^\pi R^2 \sin\theta d\theta d\phi = 4\pi R^2.$$

This surface element appears constantly in antenna radiation and far-field power calculations.

Choose coordinates to match the geometry

Rectangular waveguide naturally suggests Cartesian coordinates. Coaxial cable and cylindrical scatterers suggest cylindrical coordinates. Isolated radiators and far-field radiation suggest spherical coordinates.

The physics is unchanged; the coordinate system is chosen to make the geometry and integration limits simpler.

Common mistake: when changing coordinate systems, changing only the limits is not enough. The differential element must also be transformed. Missing ρ or $r^2 \sin\theta$ is one of the most common errors in EM integration.

10. The integral theorems, solver meaning and degree-level checkpoint

Three major theorems connect local differential behaviour with accumulated integral behaviour. You do not need to master their proofs yet, but you should understand what each one connects.

The Fundamental Theorem of Calculus connects a derivative with accumulation along one dimension:

$$\int_a^b \frac{dF}{dx} dx = F(b) - F(a).$$

The **divergence theorem** connects behaviour inside a volume to flux through its closed boundary:

$$\iiint_V (\nabla \cdot \mathbf{A}) dV = \oiint_S \mathbf{A} \cdot d\mathbf{S}.$$

The **Stokes theorem** connects local circulation over a surface to circulation around its boundary:

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}.$$

These theorems are why Maxwell's equations can be written in both integral and differential form. They are not two different theories: they are two mathematically equivalent views of the same field laws under appropriate smoothness conditions.

From calculus to a numerical EM solver

A numerical solver turns continuous operations into finite approximations. A derivative may become a difference such as

$$\frac{\partial f}{\partial x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

An integral may become a weighted sum,

$$\int f(x) dx \approx \sum_n f_n w_n.$$

In your 2D Method of Moments solver, an integral of a Green's function and basis function becomes a matrix element such as

$$Z_{mn} \sim \int_{C_m} \int_{C_n} G(\mathbf{r}, \mathbf{r}') f_m(\mathbf{r}) f_n(\mathbf{r}') dl' dl.$$

The calculus in this document is therefore not background decoration: it is the language from which the solver matrix is constructed.

A compact decision guide

Use an **ordinary derivative** when a function has one independent variable. Use a **partial derivative** when selecting one variable from several. Use an **indefinite integral** to find an antiderivative. Use a **definite integral** to accumulate over stated limits. Use a **line integral** along a path, a **surface integral** over an area in space, and a **volume integral** throughout a region. A circle on the integral sign means the path or surface is **closed**.

Degree-level checkpoint

You should now be able to explain, without relying on memorised wording:

1. Why $\partial / \partial x$ is different from d / dx .
2. Why $\mathbf{E} \cdot d\mathbf{l}$ appears in a voltage line integral.
3. Why $\mathbf{D} \cdot d\mathbf{S}$ measures flux through a surface.
4. Why reversing a line-integral path reverses its sign.
5. Why a closed surface normally uses the outward normal.
6. Why $dV = \rho d\rho d\phi dz$ contains the factor ρ .
7. How the units of an integrand and differential element predict the units of the result.
8. What the divergence theorem connects physically.
9. What Stokes' theorem connects physically.
10. Why numerical EM solvers replace derivatives and integrals with finite differences, quadrature and matrix operations.

Teach-back test: take each of the symbols dx , $d\mathbf{l}$, $d\mathbf{S}$, $d\mathbf{S}$ and dV and explain both its geometry and its units. If you can do that clearly, you have acquired one of the most useful habits for reading Maxwell's equations correctly.