

Vector Algebra and Coordinate Systems

Electromagnetic Theory Course — Mathematical Foundations — Document 3 of 23

1. Why electromagnetics is vector mathematics

Electromagnetics is not merely a subject in which quantities have numerical values. The electric field, magnetic field, current density, power flow and many other quantities also have **direction**. A scalar such as temperature needs one number at each point; a vector field such as the electric field needs both magnitude and direction at each point.

A scalar is written without bold type, for example

$$V = 12 \text{ V}, \quad T = 293 \text{ K}.$$

A vector is written in bold type:

$$\mathbf{E}, \quad \mathbf{H}, \quad \mathbf{J}, \quad \mathbf{S}.$$

Its magnitude is a scalar:

$$E = |\mathbf{E}|.$$

This distinction is fundamental. The statement $E = 5 \text{ V/m}$ says how large the electric field is. The statement

$$\mathbf{E} = 5\hat{x} \text{ V/m}$$

also says that it points in the positive x direction.

In Cartesian coordinates a vector can be resolved into three perpendicular components:

$$\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}.$$

The symbols $\hat{x}$, $\hat{y}$ and $\hat{z}$ are **unit vectors**. Each has magnitude 1 and carries direction but no physical units.

For example,

$$\mathbf{E} = 3\hat{x} + 4\hat{y} \text{ V/m}$$

has magnitude

$$|\mathbf{E}| = \sqrt{3^2 + 4^2} = 5 \text{ V/m}.$$

A field is a vector that can vary with position and time:

$$\mathbf{E} = \mathbf{E}(x, y, z, t).$$

Thus the vector at one location can point in a different direction from the vector at another location. When an EM solver draws arrows throughout a region, each arrow is a sample of the same vector field at a different position.

Teaching idea: distinguish the field from its drawing. The arrows are not physical objects filling space; they are a graphical representation of a mathematical function that assigns a vector to every point.

The three questions to ask about any vector

When you meet a vector quantity, ask: **what is its magnitude, what is its direction, and what physical units does it carry?**

Keeping those three questions separate prevents many later errors with field components, boundary conditions and power flow.

2. Components, unit vectors and position vectors

A vector can be described geometrically by an arrow or algebraically by its components. In Cartesian coordinates,

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}.$$

The magnitude follows from Pythagoras:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

A **unit vector in the direction of $\mathbf{A}$** is obtained by dividing by the magnitude:

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|}.$$

For

$$\mathbf{A} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 6\hat{\mathbf{z}},$$

we have

$$|\mathbf{A}| = \sqrt{4 + 9 + 36} = 7,$$

so

$$\hat{\mathbf{a}} = \frac{2}{7}\hat{\mathbf{x}} - \frac{3}{7}\hat{\mathbf{y}} + \frac{6}{7}\hat{\mathbf{z}}.$$

Check that $|\hat{\mathbf{a}}| = 1$.

The **position vector** locates a point relative to the coordinate origin:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

In computational electromagnetics it is essential to distinguish an **observation point $\mathbf{r}$** from a **source point $\mathbf{r}'$** . Their separation vector is

$$\mathbf{R} = \mathbf{r} - \mathbf{r}',$$

with separation distance

$$R = |\mathbf{R}|,$$

and direction

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R}.$$

This notation appears constantly in antenna theory and Green's functions. For example, the 3D free-space Green's function depends on $R = |\mathbf{r} - \mathbf{r}'|$ because propagation depends on the distance between source and observation points.

Direction cosines

If a unit vector makes angles α , β and γ with the positive x , y and z axes, then

$$\hat{\mathbf{a}} = \cos\alpha \hat{\mathbf{x}} + \cos\beta \hat{\mathbf{y}} + \cos\gamma \hat{\mathbf{z}},$$

and because its magnitude is one,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

This is useful whenever a field or current is specified by orientation rather than by components.

Components are coordinate-dependent; the vector is not

The same physical vector has different numerical components if the coordinate axes are rotated. The vector itself has not changed. This distinction becomes important when moving between Cartesian, cylindrical and spherical coordinates: **components change with the coordinate system, but the physical field does not.**

3. Vector addition, subtraction, scaling and projection

Vectors add by adding corresponding components. If

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$$

and

$$\mathbf{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}},$$

then

$$\mathbf{A} + \mathbf{B} = (A_x + B_x) \hat{\mathbf{x}} + (A_y + B_y) \hat{\mathbf{y}} + (A_z + B_z) \hat{\mathbf{z}}.$$

Geometrically, vector addition is the head-to-tail rule. In electromagnetics this represents **superposition**: if two sources independently produce fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in a linear medium, the total field is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2.$$

Subtraction is addition of the negative vector:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

Multiplying a vector by a scalar changes its magnitude and, if the scalar is negative, reverses its direction:

$$-2\mathbf{A}$$

has twice the magnitude of $\mathbf{A}$ and points oppositely.

Resolving a vector parallel and perpendicular to a direction

Suppose $\hat{\mathbf{n}}$ is a unit vector normal to a material boundary. Any vector $\mathbf{A}$ can be decomposed into normal and tangential parts:

$$\mathbf{A} = \mathbf{A}_n + \mathbf{A}_t.$$

The scalar normal component is

$$A_n = \mathbf{A} \cdot \hat{\mathbf{n}},$$

and the normal vector component is

$$\mathbf{A}_n = (A_n) \hat{\mathbf{n}}.$$

Therefore the tangential part is

$$\mathbf{A}_t = \mathbf{A} - (\mathbf{A} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}.$$

This decomposition is central to electromagnetic boundary conditions. At an interface we repeatedly ask what happens to the **normal component** of $\mathbf{D}$ or $\mathbf{B}$, and what happens to the **tangential component** of $\mathbf{E}$ or $\mathbf{H}$.

Example

Let

$$\mathbf{E} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} \text{ V/m}$$

and let the boundary normal be

$$\hat{\mathbf{n}} = \hat{\mathbf{y}}.$$

Then

$$\mathbf{E}_n = (\mathbf{E} \cdot \hat{\mathbf{y}}) \hat{\mathbf{y}} = 4\hat{\mathbf{y}} \text{ V/m},$$

and

$$\mathbf{E}_t = 3\hat{\mathbf{x}} \text{ V/m}.$$

The apparently simple act of “taking the component normal to a surface” is therefore a projection operation. The dot product gives the amount of one vector lying along another direction.

Common mistake: do not confuse a component such as E_x with the vector $E_x \hat{\mathbf{x}}$. The first is a scalar; the second is a vector.

4. The dot product: alignment, projection, work and flux

The **dot product** of two vectors is a scalar:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta,$$

where θ is the angle between them. In Cartesian components,

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

The dot product measures **alignment**. It is maximum and positive when the vectors point in the same direction, zero when they are perpendicular, and negative when they point more than 90° apart.

For the Cartesian unit vectors,

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1,$$

while

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0.$$

This is another way of saying that the Cartesian axes are orthonormal.

Projection

The scalar projection of $\mathbf{A}$ onto a unit vector $\hat{\mathbf{u}}$ is

$$A_u = \mathbf{A} \cdot \hat{\mathbf{u}}.$$

If $\hat{\mathbf{u}}$ is not a unit vector, divide appropriately by its magnitude. The vector projection is

$$\text{proj}_{\hat{\mathbf{u}}} \mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}}.$$

Why the dot product appears in line and surface integrals

If a small displacement is $d\mathbf{l}$, then

$$\mathbf{E} \cdot d\mathbf{l}$$

extracts the part of the electric field along that path. This is why potential difference involves a line integral of $\mathbf{E} \cdot d\mathbf{l}$.

If a small oriented surface has vector area

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

then

$$\mathbf{D} \cdot d\mathbf{S} = D \cos \theta dS$$

extracts the part of $\mathbf{D}$ passing normally through the surface. That is why electric flux is expressed using a dot product.

Similarly, if $\mathbf{S}$ denotes the Poynting vector, the power crossing a surface is

$$P = \iint_A \mathbf{S} \cdot d\mathbf{S}.$$

Only the component of power flow normal to the surface contributes to power crossing it.

Orthogonality in modes

In waveguides and cavity resonators, different modes often satisfy orthogonality relationships expressed by integrals containing dot products. The geometric idea is the same: the dot product detects overlap or alignment between vector patterns.

Read the symbol aloud: $\mathbf{A} \cdot \mathbf{B}$ can be read as “A dot B.” Physically, ask “how much of A lies along B?” This interpretation is more useful than memorising the component formula alone.

5. The cross product: orientation, circulation and power flow

The **cross product** of two vectors is another vector:

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. Its direction is determined by the right-hand rule.

For Cartesian unit vectors,

$$\begin{aligned}\hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}}.\end{aligned}$$

Reversing the order reverses the sign:

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}).$$

Thus the cross product is **not commutative**.

In Cartesian components, the cross product is

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \hat{\mathbf{x}} + (A_z B_x - A_x B_z) \hat{\mathbf{y}} + (A_x B_y - A_y B_x) \hat{\mathbf{z}}.$$

Why electromagnetics uses cross products so often

For an electromagnetic wave, electric field, magnetic field and propagation direction form an oriented triad. In a simple plane wave,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

points in the direction of energy flow.

For example, if

$$\mathbf{E} = E_0 \hat{\mathbf{x}}, \quad \mathbf{H} = H_0 \hat{\mathbf{y}},$$

then

$$\mathbf{S} = E_0 H_0 \hat{\mathbf{z}}.$$

The cross product is therefore encoding a physical orientation: $\mathbf{E}$, $\mathbf{H}$ and power flow have a handed relationship.

Area and triple products

The magnitude $|\mathbf{A} \times \mathbf{B}|$ equals the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$. The scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

gives the signed volume of the parallelepiped formed by the three vectors.

The vector triple-product identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

appears later when deriving the electromagnetic wave equation from Maxwell's equations.

Common mistake: the dot product produces a scalar; the cross product produces a vector. If your expected physical quantity has a direction, that immediately tells you which operation might be appropriate.

6. Cartesian coordinates and differential geometry

Cartesian coordinates are the natural choice for rectangular geometry. A point is described by (x, y, z) , and the basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are mutually perpendicular and **do not change direction with position**.

The position vector is

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

A differential displacement is

$$d\mathbf{l} = \hat{\mathbf{x}} dx + \hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz.$$

The differential volume is

$$dV = dx dy dz.$$

The oriented differential surface elements normal to the coordinate axes are

$$d\mathbf{S}_x = \hat{\mathbf{x}} dy dz,$$

$$d\mathbf{S}_y = \hat{\mathbf{y}} dx dz,$$

$$d\mathbf{S}_z = \hat{\mathbf{z}} dx dy.$$

These apparently elementary expressions are the geometric building blocks of Maxwell's integral equations.

Example: flux through a rectangular face

Suppose

$$\mathbf{D} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} \text{ C/m}^2$$

and a rectangular surface of area $A = 0.02 \text{ m}^2$ has normal $\hat{\mathbf{x}}$. The electric flux through it is

$$\Psi = \iint_A \mathbf{D} \cdot d\mathbf{S}.$$

Because $d\mathbf{S} = \hat{\mathbf{x}} dS$,

$$\mathbf{D} \cdot \hat{\mathbf{x}} = 2 \text{ C/m}^2,$$

so for a uniform field,

$$\Psi = 2A = 0.04 \text{ C}.$$

The $3\hat{\mathbf{y}}$ component contributes nothing because it lies parallel to the surface.

Why Cartesian coordinates suit rectangular waveguides

A rectangular waveguide has conducting walls at fixed values of x and y . The geometry and boundary conditions therefore align naturally with Cartesian coordinates. Modal field expressions such as

$$\sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

are a consequence of that geometry.

Coordinate choice should simplify the **geometry first**. Maxwell's equations are physical laws and do not prefer Cartesian coordinates; we choose the coordinates that make the boundaries and symmetry easiest to describe.

7. Cylindrical coordinates: wires, coax and rotational symmetry

Cylindrical coordinates describe a point using (ρ, ϕ, z) , where ρ is distance from the z axis, ϕ is azimuth angle around that axis, and z is height. The relationship with Cartesian coordinates is

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z.$$

Conversely,

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \text{atan2}(y, x).$$

The unit vectors are $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$. Unlike Cartesian basis vectors, $\hat{\rho}$ and $\hat{\phi}$ **change direction as ϕ changes**:

$$\hat{\rho} = \cos \phi \hat{x} + \sin \phi \hat{y},$$

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}.$$

This positional dependence of the basis vectors is one reason vector calculus expressions look different in cylindrical coordinates.

A differential displacement is

$$d\mathbf{l} = \hat{\rho} d\rho + \hat{\phi} \rho d\phi + \hat{z} dz.$$

The factor ρ in the azimuthal term has a simple geometric meaning: an angular change $d\phi$ sweeps an arc length $\rho d\phi$.

The differential volume is

$$dV = \rho d\rho d\phi dz.$$

Representative oriented area elements are

$$d\mathbf{S}_\rho = \hat{\rho} \rho d\phi dz,$$

$$d\mathbf{S}_\phi = \hat{\phi} d\rho dz,$$

$$d\mathbf{S}_z = \hat{z} \rho d\rho d\phi.$$

RF example: coaxial cable

For an ideal coaxial transmission line, symmetry means the electric field is radial and depends only on ρ :

$$\mathbf{E}(\rho) = E_\rho(\rho) \hat{\rho}.$$

The magnetic field circles the centre conductor:

$$\mathbf{H}(\rho) = H_\phi(\rho) \hat{\phi}.$$

Therefore

$$\mathbf{E} \times \mathbf{H}$$

points along $+\hat{z}$ for forward power flow because

$$\hat{\rho} \times \hat{\phi} = \hat{z}.$$

The coordinate system mirrors the physics so closely that the field directions become almost self-describing.

8. Spherical coordinates: radiation and angular fields

Spherical coordinates describe a point by (r, θ, ϕ) . We will use the common engineering convention in which r is radial distance, θ is the polar angle measured downward from the positive z axis, and ϕ is the azimuth measured in the x - y plane from the positive x axis.

The Cartesian relationships are

$$\begin{aligned}x &= r \sin \theta \cos \phi, \\y &= r \sin \theta \sin \phi, \\z &= r \cos \theta.\end{aligned}$$

The unit vectors are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$. All three generally change direction as the observation point moves.

A differential displacement is

$$d\mathbf{l} = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\theta}} r d\theta + \hat{\boldsymbol{\phi}} r \sin \theta d\phi.$$

The corresponding differential volume is

$$dV = r^2 \sin \theta dr d\theta d\phi.$$

For a spherical surface of constant radius r , the outward vector area element is

$$d\mathbf{S} = \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi.$$

This expression is fundamental in antenna theory because radiated power is found by integrating the radial Poynting vector over a sphere:

$$P_{\text{rad}} = \iint_{4\pi} \mathbf{S} \cdot d\mathbf{S}.$$

Why far-field antenna patterns use θ and ϕ

Far from a localized antenna, propagation is approximately radial. The electric and magnetic fields are transverse to $\hat{\mathbf{r}}$ and are naturally described by angular components such as E_θ and E_ϕ .

A common far-field structure is

$$\mathbf{E}(r, \theta, \phi) \approx \frac{e^{-jkr}}{r} [F_\theta(\theta, \phi)\hat{\boldsymbol{\theta}} + F_\phi(\theta, \phi)\hat{\boldsymbol{\phi}}].$$

The factor $1/r$ describes spherical spreading, e^{-jkr} describes phase propagation under the chosen convention, and the functions F_θ and F_ϕ describe the angular radiation pattern.

Convention warning: mathematics and physics texts do not all define θ and ϕ in the same way. Always inspect the author's coordinate diagram or definitions before using a spherical-coordinate formula.

9. Transforming components and choosing the right coordinates

A physical vector does not depend on our coordinate labels, but its **components do**. Suppose a vector lies in the x - y plane:

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}.$$

Using the cylindrical basis relationships, its cylindrical components are

$$A_\rho = \mathbf{A} \cdot \hat{\boldsymbol{\rho}} = A_x \cos\phi + A_y \sin\phi,$$

$$A_\phi = \mathbf{A} \cdot \hat{\boldsymbol{\phi}} = -A_x \sin\phi + A_y \cos\phi.$$

Notice the method: **take dot products with the new unit vectors**. This is often safer than memorising transformation matrices.

Conversely,

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi,$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi.$$

Coordinate choice by symmetry

A good coordinate system makes the geometry and expected field dependence simple:

Geometry / problem	Natural coordinates	Typical reason
Rectangular waveguide, box cavity	Cartesian	boundaries at fixed x, y, z
Infinite wire, coax, circular cylinder	Cylindrical	rotational symmetry about an axis
Isolated source, antenna far field	Spherical	outward radial propagation

This is not merely aesthetic. The correct symmetry can reduce a three-dimensional field problem to one or two variables.

For an infinite line charge on the z axis, symmetry implies

$$\mathbf{E} = E_\rho(\rho) \hat{\boldsymbol{\rho}};$$

there can be no preferred ϕ direction and no preferred z position. In Cartesian coordinates the same field is possible to write, but much less transparent.

Coordinate singularities are usually not physical singularities

At $\rho = 0$, the cylindrical angle ϕ is undefined. At the poles $\theta = 0$ and π , spherical azimuth ϕ is undefined. These are **coordinate singularities**. The physical field may remain perfectly finite there.

By contrast, the idealized field of a point charge diverging as $1/r^2$ at $r = 0$ is a physical/model singularity of the ideal point-source description.

One practical rule for EM solvers

Mesh and basis geometry may be stored in Cartesian coordinates even when the analytical physics is most naturally cylindrical or spherical. Do not confuse the solver's internal coordinate storage with the best coordinate system for understanding the field.

10. Synthesis, common mistakes and degree-level checkpoint

You should now be able to read a vector expression and identify both its algebra and its geometry. The central ideas are:

- a vector has magnitude and direction; its components depend on the chosen basis;
- unit vectors carry direction but no physical units;
- the dot product measures alignment and produces a scalar;
- the cross product measures oriented perpendicularity and produces a vector;
- normal and tangential components are projections;
- Cartesian, cylindrical and spherical coordinates describe the same physical space using different bases;
- differential line, area and volume elements inherit scale factors from the geometry;
- coordinate choice should exploit symmetry and boundary shape.

Common mistakes to eliminate

- 1. Confusing magnitude with vector.** E and $\mathbf{E}$ are not interchangeable.
- 2. Forgetting units.** Unit vectors are dimensionless, but the vector coefficient retains the physical units.
- 3. Reversing a cross product.** In general $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$; the two differ by a minus sign.
- 4. Treating cylindrical or spherical unit vectors as globally fixed.** Their directions depend on position.
- 5. Using the wrong spherical-angle convention.** Define θ and ϕ before calculating.
- 6. Thinking coordinates are physics.** A change of coordinates changes the description, not the electromagnetic field itself.

Teach-back questions

1. Explain to a first-year student the difference between E_x and $\mathbf{E}$.
2. Why does $\mathbf{A} \cdot \mathbf{B} = 0$ imply perpendicularity when both vectors are nonzero?
3. Explain physically why power flow uses $\mathbf{E} \times \mathbf{H}$ rather than $\mathbf{E} \cdot \mathbf{H}$.
4. Why is $\rho d\phi$ a length in cylindrical coordinates?
5. Why does a spherical surface element contain $r^2 \sin\theta$?
6. A field is radial around a long wire. Which coordinate system would you choose, and why?
7. Explain the difference between a coordinate singularity and a physical singularity.

Degree-level checkpoint

Let

$$\mathbf{A} = 2\hat{\mathbf{x}} - \hat{\mathbf{y}} + 2\hat{\mathbf{z}}, \quad \mathbf{B} = \hat{\mathbf{x}} + 3\hat{\mathbf{y}} - 2\hat{\mathbf{z}}.$$

Calculate $|\mathbf{A}|$, $\mathbf{A} \cdot \mathbf{B}$, the angle between the vectors, and $\mathbf{A} \times \mathbf{B}$. Then find the component of $\mathbf{A}$ normal to the plane whose unit normal is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}).$$

Finally, express the Cartesian unit vector $\hat{\mathbf{x}}$ in the cylindrical basis at $\phi = 30^\circ$.

What comes next

Document 4 develops the calculus needed to make these vectors vary through space: partial differentiation, definite and indefinite integration, and line, surface, volume and closed integrals. That prepares us to understand **gradient, divergence and curl**, and ultimately Maxwell's equations.

Mastery target: you are ready to move on when you can look at a vector formula and explain its direction, units, coordinate basis and geometric meaning without relying on memorised words alone.