

Complex Numbers, Exponentials and Phasors

Electromagnetic Theory Course — Mathematical Foundations — Document 2 of 23

1. Why electromagnetics uses complex numbers

A radio-frequency field is physically real: a voltmeter, antenna or field probe ultimately responds to a real voltage, current or field. Yet RF engineers deliberately describe sinusoidal quantities with **complex numbers**. This is not because the fields themselves are imaginary. It is because complex notation stores **amplitude and phase in one object** and turns repeated trigonometric differentiation into simple algebra.

By the end of this document you should be able to manipulate complex numbers confidently; move between rectangular, polar and exponential forms; explain Euler's identity; convert sinusoids to and from phasors; use the replacement $\partial/\partial t \rightarrow j\omega$; distinguish peak and RMS phasors; calculate impedance and complex power; and read propagation factors without losing track of the sign convention.

The imaginary unit used in electrical engineering is

$$j^2 = -1.$$

Mathematicians commonly write i . Electrical engineers use j because i is already used for current.

A complex number has the form

$$z = a + jb,$$

where a is the **real part** and b is the **imaginary part**:

$$\operatorname{Re}\{z\} = a, \quad \operatorname{Im}\{z\} = b.$$

For example,

$$z = 3 + j4.$$

This does **not** mean that $j4$ is somehow less physical than 3. The pair (3,4) is simply being encoded in a convenient algebraic form.

Complex addition is component-by-component:

$$(3 + j4) + (2 - j) = 5 + j3.$$

Multiplication follows ordinary algebra together with $j^2 = -1$:

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc).$$

A useful RF interpretation is to think of the real and imaginary axes as two perpendicular bookkeeping axes. Later, those two axes will encode the in-phase and quadrature parts of a sinusoid.

Do not say “imaginary means unreal.” In RF, the complex quantity is a mathematical representation. The measurable field is recovered by taking an appropriate real part after the calculation.

A first habit worth developing

Whenever you see a complex quantity, ask two questions:

1. What physical quantity does it represent: voltage, current, impedance, field, propagation constant?
2. Is the author using **peak** values or **RMS** values?

That second question matters because it changes power formulas by a factor of two.

2. The complex plane, magnitude, phase and conjugation

A complex number can be drawn on an **Argand diagram**. The horizontal axis represents the real part and the vertical axis represents the imaginary part. Thus $z = a + jb$ can also be regarded as a two-dimensional vector from the origin to the point (a, b) . Its magnitude is

$$|z| = \sqrt{a^2 + b^2},$$

and its phase angle is

$$\arg z = \theta = \text{atan2}(b, a).$$

The two-argument function atan2 is preferable to simply writing $\tan^{-1}(b/a)$ because it determines the correct quadrant.

For

$$z = 3 + j4,$$

we obtain

$$|z| = 5, \quad \theta \approx 53.13^\circ.$$

So the same complex number may be written in polar form as

$$z = 5 \angle 53.13^\circ.$$

Rectangular form is usually best for **addition and subtraction**. Polar or exponential form is usually best for **multiplication and division**.

If

$$z_1 = r_1 \angle \theta_1, \quad z_2 = r_2 \angle \theta_2,$$

then

$$z_1 z_2 = r_1 r_2 \angle (\theta_1 + \theta_2),$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2).$$

This is one reason RF calculations become cleaner in complex form: cascaded phase shifts simply add.

The **complex conjugate** of

$$z = a + jb$$

is

$$\bar{z} = a - jb.$$

Many RF texts write the conjugate using a superscript asterisk (read aloud as “z star”). The overbar notation used here means exactly the same operation and renders especially clearly in equations.

Two identities occur constantly in electromagnetics:

$$z \bar{z} = |z|^2$$

and

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

For example,

$$\frac{1}{3 + j4} = \frac{3 - j4}{25} = 0.12 - j0.16.$$

The conjugate is also central to **average RF power**. Later we will meet expressions such as $V \bar{I}$ and $\mathbf{E} \times \bar{\mathbf{H}}$. The conjugate is not decorative: it makes the resulting real part correspond to average power flow.

Phase is relative

An absolute phase angle has meaning only relative to a chosen time reference. If every phasor in a problem is rotated by the same phase, the physical phase differences are unchanged. This is why one voltage or field is often chosen as the 0° reference.

3. Euler's identity: the bridge between rotation and sinusoids

The central identity behind phasor analysis is

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

This is **Euler's identity** in its working RF form. It says that a complex exponential of unit magnitude corresponds to a point rotating around the unit circle as θ changes.

Replacing θ by $-\theta$ gives

$$e^{-j\theta} = \cos\theta - j\sin\theta.$$

Adding and subtracting these equations produces two extremely useful identities:

$$\begin{aligned}\cos\theta &= \frac{e^{j\theta} + e^{-j\theta}}{2}, \\ \sin\theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j}.\end{aligned}$$

These equations show that an ordinary sinusoid can be represented using rotating complex exponentials.

The exponential form of a general complex number is

$$z = re^{j\theta},$$

where $r = |z|$. Therefore

$$3 + j4 = 5e^{j0.9273}.$$

Here $53.13^\circ = 0.9273$ rad. The argument of an exponential is dimensionless and should be treated in radians when used in calculus. Engineers often display phasor angles in degrees, but differentiation formulas involving ωt assume radians.

Why differentiation becomes easy

Consider

$$e^{j\omega t}.$$

Differentiating gives

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}.$$

Differentiating again gives

$$\frac{d^2}{dt^2}e^{j\omega t} = (j\omega)^2 e^{j\omega t} = -\omega^2 e^{j\omega t}.$$

A sinusoid differentiated in ordinary trigonometric form changes from cosine to sine and accumulates sign changes. In exponential form the waveform is unchanged; it is simply multiplied by $j\omega$.

Multiplication by j corresponds to a $+90^\circ$ rotation:

$$j = e^{j\pi/2} = 1\angle 90^\circ.$$

Therefore differentiation of a sinusoid under the $e^{j\omega t}$ convention introduces both a magnitude factor ω and a $+90^\circ$ phase shift.

Similarly,

$$\int e^{j\omega t} dt = \frac{1}{j\omega} e^{j\omega t},$$

so sinusoidal integration introduces a $1/\omega$ magnitude factor and a -90° phase shift.

This one algebraic property is a major reason complex notation dominates AC circuit theory, RF engineering and frequency-domain electromagnetics.

4. From a real sinusoid to a phasor

Suppose a physical voltage is

$$v(t) = V_0 \cos(\omega t + \phi).$$

Using Euler's identity,

$$v(t) = \operatorname{Re}\{V_0 e^{j\phi} e^{j\omega t}\}.$$

The factor $e^{j\omega t}$ is common to every sinusoidal quantity at the same angular frequency. We therefore suppress it during the calculation and define the **peak phasor**

$$\tilde{V} = V_0 e^{j\phi} = V_0 \angle \phi.$$

The physical voltage is reconstructed by

$$v(t) = \operatorname{Re}\{\tilde{V} e^{j\omega t}\}.$$

A phasor is therefore **not the complete time-domain signal**. It represents the amplitude and phase of one sinusoidal frequency after the common time factor has been removed.

For example,

$$v(t) = 10 \cos(2\pi 10^9 t - 30^\circ) \text{ V}$$

has the peak phasor

$$\tilde{V} = 10 \angle (-30^\circ) \text{ V}.$$

The frequency is still part of the problem even though it does not appear inside the phasor itself.

Peak phasors and RMS phasors

Electrical engineering texts use two common conventions. The peak phasor above has magnitude V_0 . An RMS phasor instead has magnitude

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}.$$

Thus the same waveform may be represented as

$$\tilde{V}_{\text{peak}} = V_0 \angle \phi$$

or

$$\tilde{V}_{\text{rms}} = \frac{V_0}{\sqrt{2}} \angle \phi.$$

Both are valid. Mixing them is not.

Adding sinusoids using phasors

Suppose

$$\begin{aligned} v_1(t) &= 10 \cos \omega t, \\ v_2(t) &= 10 \cos(\omega t + 90^\circ). \end{aligned}$$

Their phasors are

$$\begin{aligned} \tilde{V}_1 &= 10 + j0, \\ \tilde{V}_2 &= j10. \end{aligned}$$

Therefore

$$\tilde{V} = 10 + j10 = 14.14 \angle 45^\circ,$$

and the resulting waveform is

$$v(t) = 14.14 \cos(\omega t + 45^\circ).$$

This example also illustrates why phasors must be converted to rectangular form before general addition: amplitudes alone do not add when the phases differ.

5. Phasor calculus and frequency-domain Maxwell equations

The real benefit of phasors appears when differential equations contain sinusoidal steady-state quantities. If

$$f(t) = \text{Re}\{\tilde{F}e^{j\omega t}\},$$

then

$$\frac{df}{dt} = \text{Re}\{j\omega\tilde{F}e^{j\omega t}\}.$$

Thus, in the phasor domain,

$$\boxed{\frac{d}{dt} \rightarrow j\omega}.$$

Likewise,

$$\boxed{\frac{d^2}{dt^2} \rightarrow -\omega^2}.$$

This replacement works only for sinusoidal steady state at one angular frequency ω . It is not a general replacement for arbitrary time dependence.

The same idea applies to electromagnetic fields. Write

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\tilde{\mathbf{E}}(\mathbf{r})e^{j\omega t}\},$$

and similarly for $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{D}$.

Faraday's law in the time domain is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Under the $e^{j\omega t}$ convention it becomes

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}.$$

Ampère-Maxwell law,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

becomes

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}.$$

Notice what has happened: **the time derivatives have become algebraic factors**. Spatial derivatives such as curl remain because the fields still vary with position.

This distinction is fundamental in frequency-domain EM solvers. A frequency-domain FEM or MoM solver normally removes explicit sinusoidal time dependence but continues to solve the spatial problem.

A simple circuit parallel

For an inductor,

$$v(t) = L \frac{di}{dt}.$$

In phasor form,

$$\tilde{V} = j\omega L \tilde{I}.$$

The differential equation has become an algebraic relation. The same conceptual simplification occurs when Maxwell's time-domain equations are transformed to the phasor domain.

Important: the tilde used here means “phasor.” Some textbooks use bold Roman type, an underline, a hat or no special mark at all. Always read the author's notation convention before assuming what a symbol means.

6. Impedance and admittance as complex quantities

For sinusoidal steady state, impedance is defined by

$$Z = \frac{\tilde{V}}{\tilde{I}}.$$

It is generally complex:

$$Z = R + jX,$$

where R is resistance and X is reactance. Under the $e^{j\omega t}$ convention,

$$\begin{aligned} Z_R &= R, \\ Z_L &= j\omega L, \\ Z_C &= \frac{1}{j\omega C} = -\frac{j}{\omega C}. \end{aligned}$$

The signs have physical meaning. An inductor has positive reactance and its voltage leads its current by 90° . A capacitor has negative reactance and its current leads its voltage by 90° .

Suppose

$$R = 50 \, \Omega, \quad L = 10 \, \text{nH}, \quad f = 1 \, \text{GHz}.$$

Then

$$\omega = 2\pi f,$$

so

$$X_L = \omega L \approx 62.83 \, \Omega.$$

For a series combination,

$$Z = 50 + j62.83 \, \Omega.$$

Its magnitude and angle are approximately

$$\begin{aligned} |Z| &= 80.3 \, \Omega, \\ \angle Z &= 51.5^\circ. \end{aligned}$$

The magnitude $|Z|$ is **not** resistance. It is the magnitude of the complex ratio between voltage and current. The angle tells you their phase difference.

The reciprocal of impedance is **admittance**:

$$Y = \frac{1}{Z} = G + jB,$$

where G is conductance and B is susceptance. Admittances add naturally for parallel branches just as impedances add naturally for series branches.

Connection to RF matching

A load such as

$$Z_L = 25 - j30 \, \Omega$$

has both resistive and reactive parts. Matching circuitry attempts to transform this complex impedance to the desired real system impedance, commonly $50 \, \Omega$ in RF work.

The Smith chart can be viewed as a geometrically transformed map of this same complex impedance or reflection-coefficient plane. The underlying mathematics is therefore still complex-number algebra.

Phasor division gives both gain and phase shift

If a network has transfer function

$$H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}},$$

then $|H|$ gives the amplitude ratio and $\arg H$ gives the phase shift. One complex number describes both effects at that frequency.

7. Complex power: why the conjugate appears

For time-domain voltage and current,

$$p(t) = v(t)i(t).$$

If both are sinusoidal, the instantaneous power oscillates as well as having an average value. Complex phasor notation separates the average real power from reactive behaviour.

Using **RMS phasors**, complex power is

$$S = V\bar{I} = P + jQ.$$

Here

- P is average real power in watts;
- Q is reactive power in vars;
- $|S|$ is apparent power in volt-amperes.

Thus

$$P = \operatorname{Re}\{S\}, \quad Q = \operatorname{Im}\{S\}.$$

If the voltage-current phase difference is ϕ , then

$$\begin{aligned} P &= V_{\text{rms}} I_{\text{rms}} \cos \phi, \\ Q &= V_{\text{rms}} I_{\text{rms}} \sin \phi. \end{aligned}$$

The power factor is $\cos \phi$.

If **peak phasors** are used instead, the formula becomes

$$S = \frac{1}{2} V\bar{I}.$$

This is why you must know whether a text is using peak or RMS phasors before applying a power equation.

Worked example

Let the RMS phasors be

$$\begin{aligned} V &= 10\angle 0^\circ \text{ V}, \\ I &= 0.1\angle (-60^\circ) \text{ A}. \end{aligned}$$

Then

$$\bar{I} = 0.1\angle (+60^\circ),$$

so

$$S = 1\angle 60^\circ \text{ VA} = 0.5 + j0.866 \text{ VA}.$$

Therefore

$$P = 0.5 \text{ W},$$

while $Q = 0.866 \text{ var}$ represents reversible energy exchange with reactive elements.

The electromagnetic analogue

For sinusoidal fields represented by **peak field phasors** $\mathbf{E}$ and $\mathbf{H}$, the time-average Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{2} \operatorname{Re}\{\mathbf{E} \times \bar{\mathbf{H}}\}.$$

The same conjugate structure appears because voltage-current power and field power flow are manifestations of the same sinusoidal energy accounting.

This formula is important in antennas, waveguides and RF power calculations. Integrating the normal component over a surface gives average transmitted power:

$$P = \iint_A \langle \mathbf{S} \rangle \cdot d\mathbf{A}.$$

8. Propagation, phase constants and the meaning of the exponential

Complex exponentials also describe how RF waves vary with **position**. Under the $e^{j\omega t}$ convention, a wave travelling in the $+z$ direction is commonly written

$$\tilde{E}(z) = E_0 e^{-\gamma z},$$

where

$$\gamma = \alpha + j\beta.$$

Therefore

$$\tilde{E}(z) = E_0 e^{-\alpha z} e^{-j\beta z}.$$

Reintroducing time gives

$$E(z, t) = \text{Re}\{E_0 e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\}.$$

If E_0 is real for simplicity,

$$E(z, t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z).$$

Now the two parts of γ have a clear physical meaning:

α attenuation constant, in Np/m,

β phase constant, in rad/m.

In a lossless medium, $\alpha = 0$. The phase constant is related to wavelength by

$$\beta = \frac{2\pi}{\lambda}.$$

A distance of one wavelength therefore changes the phase by 2π radians or 360° .

How to determine the direction of travel

For

$$\cos(\omega t - \beta z),$$

hold the phase constant:

$$\omega t - \beta z = C.$$

Then

$$z = \frac{\omega}{\beta} t - \frac{C}{\beta}.$$

As time increases, the point of constant phase moves toward increasing z . The wave therefore travels in the $+z$ direction.

This is safer than memorising a sign rule without understanding it.

Decibels and attenuation constants

If field amplitude decays as $e^{-\alpha z}$, then the amplitude ratio in dB is

$$20\log_{10}(e^{-\alpha z}) \approx -8.686\alpha z \text{ dB}.$$

Power falls as the square of field amplitude, but the standard $10\log_{10}$ power formula gives the same numerical attenuation in dB.

Key idea: a complex exponential can encode attenuation and phase progression simultaneously. The real part of the exponent controls exponential growth or decay; the imaginary part controls phase rotation.

9. Sign conventions: $e^{j\omega t}$ versus $e^{-j\omega t}$

You will encounter both of the following conventions in EM textbooks and software:

$$e^{j\omega t}$$

and

$$e^{-j\omega t}.$$

Neither convention is more physically correct. They are two consistent mathematical descriptions of the same real sinusoidal physics. Problems arise only when formulas derived under one convention are mixed with formulas derived under the other.

With the convention used throughout this course,

$$e^{j\omega t},$$

time differentiation becomes

$$\frac{\partial}{\partial t} \rightarrow j\omega.$$

Hence Faraday's law becomes

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}.$$

If instead the author assumes $e^{-j\omega t}$, then

$$\frac{\partial}{\partial t} \rightarrow -j\omega,$$

and the phasor form of Faraday's law becomes

$$\nabla \times \tilde{\mathbf{E}} = +j\omega \tilde{\mathbf{B}}.$$

Likewise, the spatial factor associated with a forward travelling wave changes sign when the time convention changes. Under $e^{j\omega t}$ we commonly write a $+z$ lossless wave as

$$\tilde{E}(z) = E_0 e^{-j\beta z}.$$

Under $e^{-j\omega t}$, the corresponding forward-wave phasor is commonly written

$$\tilde{E}(z) = E_0 e^{+j\beta z}.$$

In both cases the recovered real field may be written as a cosine with phase $\omega t - \beta z$.

A reliable three-step sign check

When reading a new text or implementing an EM formula:

1. Find the assumed time factor, usually near the beginning: $e^{+j\omega t}$ or $e^{-j\omega t}$.
2. Check one derivative-based equation, such as Faraday's law.
3. Reconstruct a real travelling wave and verify that the phase moves in the intended direction.

Do not copy a propagation factor from one convention and an impedance or Maxwell equation from another without checking the signs.

Frequency-domain software

Many numerical solvers never explicitly display the suppressed time factor. The convention may instead be stated in the manual. It affects the sign of imaginary material parameters, radiation conditions, Green's functions and propagation factors. For computational electromagnetics, this is therefore not merely typographical preference.

Teaching point: signs in complex EM formulas are often convention-dependent; measurable quantities such as power, field magnitude and propagation direction are not. Always separate convention from physics.

10. Synthesis, common mistakes and degree-level checkpoint

Complex notation should now be seen as a compression system for sinusoidal RF physics. A phasor such as

$$\vec{E} = 20\angle(-35^\circ) \text{ V/m}$$

encodes the field amplitude and its phase relative to a reference. Under the $e^{j\omega t}$ convention the corresponding real field component is

$$E(t) = 20\cos(\omega t - 35^\circ) \text{ V/m.}$$

Differentiation is represented by multiplication by $j\omega$; spatial phase progression can be represented by $e^{-j\beta z}$; loss and phase can be combined into $e^{-\gamma z}$; and average power involves a conjugate product.

The essential chain of ideas

$$\begin{aligned} a + jb &\Leftrightarrow r\angle\theta \Leftrightarrow re^{j\theta}. \\ e^{j\theta} &= \cos\theta + j\sin\theta. \\ x(t) &= \text{Re}\{\tilde{x}e^{j\omega t}\}. \\ \frac{d}{dt} &\rightarrow j\omega. \\ Z &= \frac{\tilde{V}}{\tilde{I}}. \\ \langle S \rangle &= \frac{1}{2} \text{Re}\{\mathbf{E} \times \overline{\mathbf{H}}\} \quad (\text{peak field phasors}). \end{aligned}$$

Common mistakes to eliminate

- Treating j as a mysterious physical quantity rather than an algebraic operator satisfying $j^2 = -1$.
- Adding phasor magnitudes while ignoring phase.
- Mixing degrees and radians inside calculus.
- Forgetting whether a phasor magnitude is peak or RMS.
- Using $S = V\tilde{I}$ with peak phasors without the required factor $1/2$.
- Forgetting the conjugate in average-power calculations.
- Calling $|Z|$ “the resistance.”
- Mixing $e^{j\omega t}$ and $e^{-j\omega t}$ sign conventions.
- Forgetting that phasor analysis assumes sinusoidal steady state at a specified frequency.

Degree-level checkpoint

1. Convert $-3 + j3$ to polar form. **Answer:** $4.243\angle 135^\circ$.
2. Evaluate $(2\angle 30^\circ)(4\angle -20^\circ)$. **Answer:** $8\angle 10^\circ$.
3. Convert $8\cos(\omega t + 25^\circ)$ to a peak phasor. **Answer:** $8\angle 25^\circ$.
4. A peak current phasor is $2\angle -40^\circ$ A. What is its RMS phasor? **Answer:** $\sqrt{2}\angle -40^\circ$ A.
5. Under the $e^{j\omega t}$ convention, what phasor operation represents a time derivative? **Answer:** multiplication by $j\omega$.
6. At 100 MHz, find the reactance of a 100 nH inductor. **Answer:** $X_L = 2\pi fL \approx 62.83 \Omega$.
7. Explain why $\vec{E}(z) = E_0 e^{-j\beta z}$ represents propagation in $+z$ when combined with $e^{j\omega t}$.
8. Explain in words why the conjugate appears in $V\tilde{I}$ and $\mathbf{E} \times \overline{\mathbf{H}}$.

Teach-back test

You are ready to move on when you can explain, without notes, **why** phasors work rather than merely how to manipulate them. In particular, you should be able to derive Euler’s relation between cosine and complex exponentials, reconstruct a real sinusoid from a phasor, explain the $j\omega$ derivative rule, distinguish peak from RMS phasors, and identify the assumed time convention in an unfamiliar EM formula.

The next foundation document develops **vector algebra and coordinate systems**. That is where complex scalar notation and spatial direction begin to come together in the language used by Maxwell’s equations.