

Mathematical Language and Notation for Electromagnetics

Electromagnetic Theory Course — Mathematical Foundations — Document 1 of 23

1. How to read the language of electromagnetics

Electromagnetics becomes much easier when the notation stops feeling like a code. The purpose of this first document is therefore not to teach Maxwell's equations yet. It is to make the *language* used in those equations familiar enough that, when the physics arrives, your attention can stay on the physics.

By the end of this document you should be able to distinguish a scalar from a vector and a field; interpret subscripts, super scripts, hats, bold type and primes; identify source and observation coordinates; read derivatives and integrals without losing track of what varies; check SI units; and translate a compact EM expression into an ordinary-language statement.

A typical EM expression might look like

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{x}} E_x(\mathbf{r}, t) + \hat{\mathbf{y}} E_y(\mathbf{r}, t) + \hat{\mathbf{z}} E_z(\mathbf{r}, t).$$

Do not try to “see” this as one object. Read it in layers:

- $\mathbf{E}$ is a **vector field**: at each position and time it has a magnitude and direction.
- $\mathbf{r}$ means position; t means time.
- E_x, E_y, E_z are **scalar component functions**.
- $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors defining directions.

Read it aloud: “The electric field at position $\mathbf{r}$ and time t equals its x-directed component plus its y-directed component plus its z-directed component.”

The central notation rule

Throughout this course, **bold symbols denote vectors** and ordinary italic symbols denote scalars. Thus

$$\mathbf{E} \text{ is a vector, whereas } E = |\mathbf{E}| \text{ is its magnitude.}$$

Textbooks differ. Some use arrows, such as $\vec{E}$, and some use bold italic. Always check a book's notation before assuming two symbols mean different things.

Four questions to ask whenever you meet an equation

1. **What kind of object is each symbol?** Scalar, vector, matrix, coordinate, operator?
2. **What does it depend on?** Position, time, frequency, material properties?
3. **What are its units?** An equation with inconsistent dimensions cannot be physically correct.
4. **What operation is being performed?** Addition, dot product, cross product, differentiation, integration?

These four questions will be used repeatedly throughout the course.

2. Scalars, vectors and fields

A **scalar** has magnitude but no direction. Examples include frequency f , time t , temperature T , electric potential V , permittivity ϵ and conductivity σ .

A **vector** has magnitude and direction. Important EM vectors include electric field $\mathbf{E}$, electric flux density $\mathbf{D}$, magnetic field strength $\mathbf{H}$, magnetic flux density $\mathbf{B}$, current density $\mathbf{J}$ and the Poynting vector $\mathbf{S}$.

Quantity	Symbol	Type	SI unit
Electric field	$\mathbf{E}$	vector	V/m
Electric flux density	$\mathbf{D}$	vector	C/m ²
Magnetic field strength	$\mathbf{H}$	vector	A/m
Magnetic flux density	$\mathbf{B}$	vector	T
Current density	$\mathbf{J}$	vector	A/m ²
Electric potential	V	scalar	V
Charge density	ρ_v	scalar	C/m ³

A vector can be resolved into components. In Cartesian coordinates,

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}.$$

Its magnitude is

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Notice the difference between A_x and $\hat{\mathbf{x}}$. The first is a scalar telling you **how much** of the vector points in the x direction. The second is a unit vector telling you **which direction** x is.

A **field** assigns a quantity to every point in some region. A scalar field assigns one scalar to every point: $V(x, y, z)$. A vector field assigns a vector to every point: $\mathbf{E}(x, y, z)$.

This distinction is fundamental. A single vector might describe the force on one charge. A vector field describes what force *would* act on a suitable test charge at every point in space.

For an electric field,

$$\mathbf{F} = q\mathbf{E},$$

so $\mathbf{E}$ may be understood operationally as force per unit positive test charge. This is not yet the full theory of $\mathbf{E}$, but it gives the symbol a physical meaning.

Magnitude is not the same thing as a component

If

$$\mathbf{E} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} \text{ V/m},$$

then $E_x = 3 \text{ V/m}$ and $E_y = 4 \text{ V/m}$, while

$$E = |\mathbf{E}| = 5 \text{ V/m}.$$

Common trap: writing $E = 3 + 4 = 7 \text{ V/m}$. Components in perpendicular directions do not add as ordinary magnitudes; the vector geometry matters.

3. Coordinates, position vectors and source/observation points

In Cartesian coordinates a point is specified by (x, y, z) . The corresponding **position vector** is

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}.$$

This notation lets us write a field compactly as $\mathbf{E}(\mathbf{r})$ instead of repeatedly writing $\mathbf{E}(x, y, z)$.

Two positions occur constantly in computational electromagnetics:

- $\mathbf{r}$ — the **observation point**, where the field is being calculated.
- $\mathbf{r}'$ — the **source point**, where charge or current is located.

The prime does not mean “differentiate”. Here it is simply a label. The displacement from source to observation point is often written

$$\mathbf{R} = \mathbf{r} - \mathbf{r}',$$

and the scalar separation is

$$R = |\mathbf{R}| = |\mathbf{r} - \mathbf{r}'|.$$

This distinction becomes central in Green’s functions and the Method of Moments. A typical kernel might depend on both positions:

$$G(\mathbf{r}, \mathbf{r}').$$

Read this as: “the response observed at $\mathbf{r}$ due to a unit source at $\mathbf{r}'$.”

Unit vectors

A hat normally denotes a unit vector. Therefore

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = |\hat{\mathbf{z}}| = 1.$$

In Cartesian coordinates these directions are fixed everywhere. In cylindrical and spherical coordinates, however, the local unit vectors change direction as position changes. That is one reason vector calculus looks more complicated in non-Cartesian coordinates.

Three coordinate systems dominate EM:

Coordinate system	Typical geometry	Typical notation
Cartesian	rectangular waveguides, layered structures	(x, y, z)
Cylindrical	wires, coax, circular guides	(ρ, ϕ, z)
Spherical	antennas, radiation, isolated sources	(r, θ, ϕ)

The important principle is to choose coordinates that respect the geometry. A circular conductor is usually simpler in cylindrical coordinates than Cartesian coordinates because the boundary becomes “constant radius”.

Direction vectors between points

The unit vector from source to observation point is

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

In radiation and scattering formulas this unit vector often defines the direction of propagation or observation.

Notation habit: when you see r , R , $\mathbf{r}$, $\mathbf{R}$ and $\hat{\mathbf{R}}$ in the same derivation, pause and identify each one explicitly. They are related but not interchangeable.

4. Functions, indices, subscripts and superscripts

A function is a rule that maps one or more independent variables to a value. In EM we routinely use functions of several variables:

$$E_x = E_x(x, y, z, t).$$

This says that the x component of electric field may vary with all three spatial coordinates and with time. If a problem is invariant in z, we may have

$$\frac{\partial E_x}{\partial z} = 0,$$

which means the field does not change as we move in the z direction.

Subscripts can mean different things

Context matters. In E_x , the subscript identifies a Cartesian component. In f_c , the subscript labels the quantity as “cutoff frequency”. In Z_0 , the zero is part of the conventional symbol for characteristic impedance. In TE_{10} , the integers are mode indices. Never assume a subscript means “component” without checking the definition.

Repeated indices and arrays

Numerical EM introduces vectors and matrices such as

$$\mathbf{Z}\mathbf{I} = \mathbf{V}.$$

The matrix element Z_{mn} can be read as “row m , column n ”. In a Method of Moments solver it often represents the tested field response at observation/testing function m due to source/basis function n .

Likewise, I_n may be the unknown coefficient multiplying basis function n .

Superscripts

Superscripts may indicate powers,

$$R^2,$$

but they may also be labels. For example, one author might use $\mathbf{E}^{\text{inc}}$ and $\mathbf{E}^{\text{scat}}$ for incident and scattered fields. These are not exponents.

Complex conjugation is often shown with a superscript asterisk; an overbar is an equivalent notation. If $Z = R + jX$, then

$$\bar{Z} = R - jX.$$

This becomes important in complex power and time-average Poynting-vector calculations.

Peak, RMS and phasor notation

Authors use different conventions for sinusoidal quantities. You may see

$$E(t) = E_0 \cos(\omega t + \phi),$$

where E_0 is peak amplitude, or an RMS phasor $\tilde{E}$ whose magnitude is already divided by $\sqrt{2}$. The factor of $1/2$ in average-power formulas depends on which convention is used.

In this course we will state the convention whenever it matters. A safe rule is: **never use a complex power formula until you know whether the phasor magnitude is peak or RMS.**

Parentheses tell you dependence

Compare

$$\epsilon E$$

with

$$\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}).$$

The second expression explicitly allows the material permittivity to vary with position. This is how an inhomogeneous medium enters the mathematics.

5. Operators and differential notation

An **operator** is an instruction to perform a mathematical operation on something. In EM the most important operator is ∇ , pronounced “del” or “nabla”. Its meaning depends on how it is combined with a field.

In Cartesian coordinates,

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}.$$

Think of this as a *vector differential operator*. It is not itself a physical field.

Three constructions recur throughout EM:

$$\begin{aligned} \nabla V & \quad \text{gradient of a scalar field,} \\ \nabla \cdot \mathbf{A} & \quad \text{divergence of a vector field,} \\ \nabla \times \mathbf{A} & \quad \text{curl of a vector field.} \end{aligned}$$

The next foundation documents will develop these carefully. For now, the notation is what matters: juxtaposition, a dot and a cross mean three different operations.

Ordinary versus partial derivatives

If y depends only on x , an ordinary derivative is appropriate:

$$\frac{dy}{dx}.$$

If E_x depends on several variables, then

$$\frac{\partial E_x}{\partial x}$$

means: vary x while holding the other independent variables fixed.

For example, if

$$E_x(x, y) = 3x^2y,$$

then

$$\frac{\partial E_x}{\partial x} = 6xy, \quad \frac{\partial E_x}{\partial y} = 3x^2.$$

The same field can therefore have different rates of change in different directions.

The Laplacian

Another important operator is

$$\nabla^2 = \nabla \cdot \nabla.$$

For a scalar V in Cartesian coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

It appears in Laplace’s equation, Poisson’s equation and the Helmholtz/wave equation.

Reading habit: identify what an operator acts on. In $\nabla \times \mathbf{E}$, the operator acts on the entire vector field $\mathbf{E}$. In $\partial \mathbf{B} / \partial t$, every component of $\mathbf{B}$ is differentiated with respect to time.

6. Differential elements and integration notation

Integration in EM is almost always integration **over a physical domain**: along a path, across a surface or throughout a volume. The differential element tells you which kind of domain is involved.

One-dimensional integrals

An indefinite integral,

$$\int f(x) dx,$$

represents a family of antiderivatives. A definite integral,

$$\int_a^b f(x) dx,$$

accumulates a quantity from a to b .

Line integrals

A line element is a small directed displacement $d\mathbf{l}$. Thus

$$\int_C \mathbf{E} \cdot d\mathbf{l}$$

accumulates the component of $\mathbf{E}$ tangent to a path C . The dot product is essential: only the field component parallel to each little path element contributes.

If the path is closed, we write

$$\oint_C \mathbf{E} \cdot d\mathbf{l}.$$

The circle on the integral sign means “return to the starting point.”

Surface integrals

A directed surface element is

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the local unit normal. A flux integral such as

$$\iint_S \mathbf{D} \cdot d\mathbf{S}$$

counts the component of $\mathbf{D}$ passing normally through the surface.

For a closed surface surrounding a volume, the notation is often

$$\oiint_S \mathbf{D} \cdot d\mathbf{S}.$$

Different books typeset the closed-surface symbol differently; the defining words matter more than the typography.

Volume integrals

A volume element in Cartesian coordinates is

$$dV = dx dy dz.$$

If ρ_v is charge density in C/m^3 , then total charge in a volume is

$$Q = \iiint_V \rho_v dV.$$

The units explain the operation:

$$\frac{\text{C}}{\text{m}^3} \times \text{m}^3 = \text{C}.$$

This kind of dimensional check is one of the most useful habits in EM.

Orientation matters

Line and surface integrals can change sign when orientation is reversed. Closed loops and surfaces are normally assigned a positive orientation using a right-hand-rule convention. This later links Stokes’ theorem to Faraday’s law and Ampère-Maxwell law.

Key idea: integration is not merely “finding an area”. In EM it means summing infinitesimal physical contributions over a specified geometric object.

7. Complex notation and the frequency-domain viewpoint

RF engineers use complex numbers because sinusoidal signals are much easier to manipulate as rotating complex exponentials than as repeated combinations of sines and cosines.

The electrical-engineering imaginary unit is

$$j = \sqrt{-1}.$$

We normally use j rather than i because i is widely used for current.

A complex number can be written in rectangular form,

$$Z = R + jX,$$

or polar/exponential form,

$$Z = |Z|e^{j\phi}.$$

Euler's identity is the bridge:

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

Therefore a physical sinusoid may be represented as the real part of a complex expression:

$$E(t) = \text{Re}\{\tilde{E}e^{j\omega t}\}.$$

Here $E(t)$ is the real, measurable time-domain field and $\tilde{E}$ is a complex phasor containing amplitude and phase.

Why the phasor is useful

Differentiation with respect to time becomes multiplication by $j\omega$:

$$\frac{d}{dt}(\tilde{E}e^{j\omega t}) = j\omega\tilde{E}e^{j\omega t}.$$

This turns differential equations in sinusoidal steady state into algebraic equations. It is one of the reasons frequency-domain EM is so compact.

Complex conjugation

If

$$Z = R + jX,$$

then

$$\bar{Z} = R - jX.$$

A product with a conjugate often produces a real magnitude:

$$Z\bar{Z} = |Z|^2.$$

The conjugate also appears in time-average power expressions such as

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re}\{\mathbf{E} \times \bar{\mathbf{H}}\}$$

when peak phasors are used.

Sign conventions

Some texts assume $e^{j\omega t}$; others use $e^{-j\omega t}$. Both are valid if used consistently. The apparent sign of propagation factors and Maxwell curl equations changes with that convention. Before comparing two formulas, check the assumed time dependence.

8. SI units, dimensions and engineering prefixes

Units are not decoration. They are part of the physics and are an excellent error detector.

Important electromagnetic quantities include:

Quantity	Symbol	SI unit	Useful equivalent
Electric field	E	V/m	N/C
Electric flux density	D	C/m ²	—
Magnetic field strength	H	A/m	—
Magnetic flux density	B	T	Wb/m ²
Permittivity	ϵ	F/m	—
Permeability	μ	H/m	—
Conductivity	σ	S/m	—
Power density	S	W/m ²	—

The constitutive relation

$$\mathbf{B} = \mu \mathbf{H}$$

must be dimensionally consistent. Since μ has units H/m and $\mathbf{H}$ has units A/m, their product has the units of tesla.

Likewise,

$$\mathbf{D} = \epsilon \mathbf{E}.$$

If an equation produces volts when you expect watts, something has gone wrong no matter how elegant the algebra looks.

Engineering prefixes

RF and EM routinely span enormous numerical ranges:

Prefix	Symbol	Factor
pico	p	10^{-12}
nano	n	10^{-9}
micro	μ	10^{-6}
milli	m	10^{-3}
kilo	k	10^3
mega	M	10^6
giga	G	10^9

Be particularly careful with case: **m** means milli, **M** means mega. A 10 mW transmitter and a 10 MW transmitter differ by a factor of one billion.

Radians are dimensionless, but still meaningful

Phase is usually expressed in radians. In

$$\cos(\omega t - \beta z),$$

the argument of the cosine must be dimensionless. Therefore ω has units rad/s, t has s, β has rad/m and z has m. Both products represent phase.

Dimensional analysis as a teaching tool

Suppose a student writes

$$P = \mathbf{E} \times \mathbf{H}.$$

The right-hand side has units W/m², so it cannot directly be total power in watts. It is a **power-flow density**. To obtain power through a surface we integrate:

$$P = \iint_S \mathbf{S} \cdot d\mathbf{S}.$$

Units often reveal the conceptual distinction before the derivation does.

9. Reading complete EM equations without getting lost

The goal now is to translate compact equations into structured statements.

Example 1: material relation

$$\mathbf{B} = \mu \mathbf{H}.$$

Read it as: “In a linear isotropic medium, magnetic flux density equals permeability times magnetic field strength.” The equation does **not** say $\mathbf{B}$ and $\mathbf{H}$ are the same physical quantity; it says they are related by a material property.

Example 2: Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

The cross product tells you immediately that $\mathbf{S}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{H}$. Its units are W/m^2 , so it represents power flow per unit area rather than total power.

Example 3: Gauss’s law

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enclosed}}.$$

Break it down:

1. S is a closed surface.
2. $d\mathbf{S}$ points locally outward.
3. The dot product takes the outward-normal component of $\mathbf{D}$.
4. The integral sums that flux over the entire closed surface.
5. The result equals the enclosed free charge.

The symbol-heavy equation is therefore a concise geometric statement.

Example 4: Faraday’s law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

The left side is circulation of electric field around a closed contour. The right side is minus the time rate of change of magnetic flux through a surface spanning that contour. The minus sign encodes Lenz’s law: the induced effect opposes the change producing it.

Example 5: travelling wave notation

$$\mathbf{E}(z, t) = \hat{\mathbf{x}} E_0 \cos(\omega t - \beta z + \phi).$$

Interpret each piece:

- $\hat{\mathbf{x}}$: field is x-polarized.
- E_0 : peak field magnitude.
- ωt : temporal phase evolution.
- $-\beta z$: phase decreases with increasing z for this convention, corresponding to $+z$ propagation.
- ϕ : initial phase offset.

Teaching principle: never present a major EM equation without first identifying its geometry, its dependent variables, its units and the meaning of each operator.

10. Consolidation, teach-back and degree-level checkpoint

This first document is successful if you can now *parse* EM notation even when you have not yet learned the complete physical law behind it.

Essential distinctions to be able to explain

Pair	Distinction
$\mathbf{E}$ vs E	vector field versus its scalar magnitude
$\mathbf{B}$ vs $\mathbf{H}$	flux density versus magnetic field strength
$\mathbf{r}$ vs $\mathbf{r}'$	observation position versus source position
$d\mathbf{l}$ vs $d\mathbf{S}$	directed line element versus directed surface element
$\oint$ vs $\oint$	open/general integration versus closed-path integration
d/dx vs $\partial/\partial x$	ordinary derivative versus partial derivative
$\cdot$ vs $\times$	dot product versus cross product
$\mathbf{E}(t)$ vs $\tilde{\mathbf{E}}$	real time-domain field versus complex phasor

Mini worked checks

1. Component/magnitude check. If $\mathbf{H} = 6\hat{\mathbf{x}} + 8\hat{\mathbf{z}}$ A/m, then

$$|\mathbf{H}| = \sqrt{6^2 + 8^2} = 10 \text{ A/m.}$$

2. Dimensional check. If ρ_v is in C/m³ and a volume element is in m³, then $\rho_v dV$ is charge in coulombs.

3. Geometry check. In $\mathbf{E} \cdot d\mathbf{l}$, only the field component tangent to the path contributes. In $\mathbf{D} \cdot d\mathbf{S}$, only the component normal to the surface contributes.

4. Source/observation check. In $G(\mathbf{r}, \mathbf{r}')$, changing $\mathbf{r}'$ moves the source; changing $\mathbf{r}$ moves the observer.

Teach-back questions

Try answering these aloud, as if teaching a first-year engineering class:

1. Why is a field more than a single vector?
2. Why does $\mathbf{E} \times \mathbf{H}$ have a direction?
3. Why do we need partial derivatives in electromagnetics?
4. What does a prime on $\mathbf{r}'$ usually mean in integral-equation EM?
5. Why is $\oint$ different from $\oint$?
6. How can units reveal that a formula represents power density rather than power?
7. What information is stored in a complex phasor that is not visible in a real amplitude alone?
8. Why must the assumed $e^{\pm j\omega t}$ convention be stated?

Degree-level checkpoint

You are ready for the next document if you can look at

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

and, even before deriving it, say confidently:

- $\mathbf{E}$ and $\mathbf{H}$ are vector fields;
- $\nabla \times$ is the curl operator;
- j indicates frequency-domain complex notation;
- ω is angular frequency;
- μ is permeability;
- both sides must have compatible dimensions;
- the equation is a local differential statement about how one field is related to another.

That ability to unpack notation is the foundation for understanding rather than memorizing Maxwell's equations.

Next document

Foundation Document 2 — Complex Numbers, Exponentials and Phasors will develop the complex mathematics used in RF engineering from first principles, including magnitude/phase, Euler's identity, sinusoidal steady state, impedance and the origin of the phasor method.